XX. *Researches in Physical Geology.* *By* W. Hopkins, *Esq. M.A. F.R.S., Fellow of the Royal Astronomical Society, of the Geological Society, and of the Cambridge Philosophical Society.—First Series.*

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On the Phenomena of Precession and Nutation, assuming the Fluidity of the Interior of the Earth.

§. *Preliminary Observations on the Refrigeration of the Globe.*

Before I proceed to the discussion of the question which forms the principal subject of the present communication, I shall offer some general remarks on the refrigeration of the globe, as introductory not only to this memoir, but to others which I hope hereafter to bring under the notice of the Society.

In the first place, we may observe that there are two distinct processes of cooling, of which one belongs to bodies which are either solid or imperfectly fluid, and is termed cooling by *conduction,* and the other to masses in that state of more perfect fluidity which admits of a free motion of the component particles among themselves. In this case the cooling is said to take place by *circulation* or *convection.* The nature of the former process has been ascertained with considerable accuracy by experiment, and the laws of the phenomena have been made the subject of mathematical investigation, but of the exact laws of cooling by the latter process we are comparatively ignorant. It is manifest, however, that since *time* must be necessary for the transmission of the hotter and lighter particles from the central to the superficial parts of the mass, as well as for that of the colder and heavier particles in the opposite direction, the temperature must increase with the depth beneath the surface; and, moreover, that this increase will be the more rapid, the more nearly the fluidity of the mass approaches that limit at which this process of cooling would cease, and that by conduction begin, since the rapidity of circulation would constantly diminish as the fluidity should approximate to that limit. But still, even in this limiting case, it seems probable that the tendency to produce an equality of temperature throughout the mass will be much greater, and consequently the rate of increase of temperature in approaching the centre much less, than if the cooling of the mass had proceeded by conduction during the same time, the conductive power being very small.

If the matter composing the globe was originally in a high state of fluidity from heat, the process of cooling would undoubtedly, in the first instance, be by circulation. The manner in which the transition will take place from this mode of refrigeration to that by conduction, depends on certain conditions, of which, in our speculations on this subject, it is important to form a distinct conception.

Since the heat increases with the distance from the surface while the mass is cooling by circulation, the tendency to solidification, so far as it depends on this cause, will be greatest at the surface and least at the centre; but, on the other hand, the pressure is least at the surface and greatest at the centre; and consequently the tendency to solidify, as depending on this cause, will be greatest at the centre and least at the surface. To estimate this tendency under the joint influence of these causes, it would be necessary, in the first place, to know the law according to which the temperature increases in descending from the surface to the centre, while the mass is cooling by circulation; and secondly, the influence of the temperature in resisting solidification, as compared with that of the pressure in promoting it. These, however, are points on which we possess at present little or no experimental evidence, and therefore the only conclusion at which we can arrive is this,—that if the augmentation of the temperature with that of the depth be so rapid, that its effect in resisting the tendency to solidify be greater than that of the increase of pressure to promote it, there will be the greatest tendency to become *imperfectly fluid,* and afterwards to solidify in the superficial portions of the mass; whereas if the effect of the augmentation of pressure predominate over that of the temperature, this transition from perfect to imperfect fluidity, and subsequent solidity, will commence at the centre.

If we suppose the former of these cases to hold, it would appear that no incrustation of the surface could take place so long as any inferior portion of the mass retained its perfect fluidity, because as the superior particles should become condensed they would continually descend into the perfect fluid beneath, always supposing the mass in that state in which an increase of specific gravity would result from a decrease of temperature. The process of circulation would thus go on till every part of the mass should have lost that degree of more perfect fluidity, which admits of a motion of the particles among themselves being excited by their unequal refrigeration. The circulation, therefore, would cease nearly contemporaneously in every part of the mass, which would then begin to cool by conduction, rapidly at the surface exposed to the low temperature of the planetary space, and extremely slowly in the central parts, on account of the small conductive power of the matter composing the earth. Consequently the globe would consist, after a certain time, of an exterior solid crust, and interior fluid matter, of which the fluidity would increase in approaching the centre, where it might still approach to that more perfect fluidity which admits of cooling by convection. With reference, however, to the mechanical action of any forces producing either motion or hydrostatic pressure in the interior mass, the whole of it might, as an approximation, be considered perfectly fluid. No attempt has yet been made to determine the present probable thickness of the earth's crust, assuming it to have been originally in a state of fluidity, on account of the difficulty already mentioned, arising from our ignorance of the influence of high temperature in resisting solidification, compared with that of great pressure in promoting it. All that has hitherto been determined on the subject is, that the present state of the earth's surface may be consistent with the existence of a solid crust, of which the thickness is small compared with the earth's radius.

Let us now recur to the other case above mentioned, that in which the increase of pressure in descending towards the centre of the mass is supposed to have a greater effect in promoting solidification than the increase of temperature in preventing it. Supposing the mass to have been first in a state in which every part was cooling by convection, this process would first cease, and that of cooling by conduction begin at the centre, while the superior portion would still continue to cool by convection, so that these two processes would for a time be going on simultaneously in different parts of the mass. It is manifest, however, that the central portion, cooling by conduction, would constantly increase, while the exterior portion, cooling by convection, would constantly diminish, so that at length no part of the mass would be cooling by the latter process. Before it should reach this stage of the refrigeration the central portion of a mass so large as the earth might become perfectly solid, so that at the instant when the circulation should entirely cease, the whole might consist of a solid central nucleus, surrounded by the external portion still in a state of fusion, and of which the fluidity would vary continuously from the solidity of the nucleus to the fluidity of the surface, where, at the instant we are speaking of, it would be just such as not to admit of circulation.

When the mass should have arrived at this stage of the cooling, a change would take place in the process of solidification, which it is important to remark. The superficial parts of the mass must in all cases cool the most rapidly, and now (in consequence of the imperfect fluidity) being no longer able to descend, a *crust* will be formed on the surface, from which the process of solidification will proceed far more rapidly downwards, than upwards on the solid nucleus. Consequently, then, our globe would arrive at that state, according to the mode of cooling we are now considering, in which it would be composed of a solid shell, and a solid central nucleus, with matter in a state of fusion between them, the fluidity of which, however, would necessarily be less than that which might exist in the fused mass very near the centre in the case previously considered.

With respect to the thickness of the shell which may be consistent with the present appearances of the earth's surface, the same conclusion will hold as in the former case, i. e. it may be small compared with the earth's radius. What would be the radius of the solid nucleus at the instant of the first incrustation of the surface, or that which would correspond to any assigned thickness of the exterior shell, it is quite impossible to determine from the want of all experimental evidence respecting the tendency of great pressure to promote solidification at very high temperatures, and our ignorance of the temperature at which the superficial incrustation of a large mass would begin, when exposed to the temperature of the planetary space. It is, therefore, manifestly impossible to decide by any such reasoning as the above, whether the exterior shell and solid nucleus are now united, or are separated by matter still in a state of fusion*.

Upon the whole, reasoning such as the above can lead us to nothing more definite than the following conclusions respecting the actual state of the earth, assuming it to have once been in a state of perfect fluidity.

(1.) It may consist of a solid exterior shell and an internal mass in a state of fusion, of which the fluidity is greatest at the centre; and it is possible that the thickness of the shell may be small compared with its radius, and the fluidity at the centre may approximate to that which would admit of cooling by convection.

(2.) It may consist of an exterior shell, and a central solid nucleus, with matter in a state of fusion between them. The thickness of the shell, as well as the radius of the solid nucleus, may possibly be small compared with the radius of the earth. The fluidity of the intervening mass must necessarily be considerably more imperfect than that which would just admit of cooling by circulation.

(3.) The earth may be solid from the surface to the centre.

It appears then that the direct investigation of the manner of the earth's refrigeration, assuming its original fluidity from heat, still leaves us in a state of perfect uncertainty as to the actual condition of its central parts, not from any imperfection in the mathematical part of the investigation, but from the want of the experimental determination of values which it must ever be found extremely difficult, if not impossible, to obtain with accuracy. Under these circumstances, we are naturally led to consider whether any other more indirect test may be found of the truth of the hypothesis of central fluidity. In reflecting on this subject, it occurred to me some time ago, that such a test might possibly be found in the delicate but well-defined phenomena of precession and nutation. The connexion between these phenomena and the interior fluidity will at once be seen by those accustomed to physical investigations of this nature; since it is manifest, that the direct action of those forces which produce the precessional motion of the earth's pole must be entirely different on the interior part of the earth, if that part be fluid, to that which must be exerted, if the interior part be solid. It becomes, therefore, a matter of interest to examine how far the internal

* M. Poisson, was, I believe, the first to advocate the hypothesis of the solidification of the earth having commenced from the centre, and has stated in general terms that, in such case, it would proceed to the surface which would be the last to solidify (Théorie de la Chaleur, p. 428.). It is manifest, however, from what has been advanced, that this could not be literally correct, but that the solidification must necessarily commence at the surface before the whole internal portion had become solid. The distinction is of little consequence as respects the object which M. Poisson had in view, but is of the highest importance with reference to Geological speculation, because it shows, that, supposing the earth once to have been fluid, it must be now or have been at some antecedent epoch in that state in which a solid exterior crust rests on an imperfectly fluid and incandescent mass beneath. It forms no part of my immediate object, to consider whether the hypothesis of this being or having been once the state of our planet, best enables us to account for the igneous matter which has been injected so generally into the sedimentary portion of the earth's crust, but it is important to know, that this state of the earth, assuming its original fluidity, is one through which it must necessarily have passed in the course of its refrigeration, whatever might be the process of its solidification.

fluidity may consist with the observed phenomena of the precessional motion of the pole. These phenomena have been shown to be perfectly in accordance with the internal solidity of the earth under certain hypotheses, which may be deemed perfectly reasonable, respecting the law of density; but so far from any attempts having been hitherto made to determine what would be the precessional motion on the supposition of interior fluidity, I am not aware that the problem has been before suggested. I shall now proceed to its solution, which forms the principal object of the present memoir.

On Precession and Nutation; assuming the Fluidity of the Interior of the Earth.

In the present memoir I shall investigate the amount of the luni-solar precession and nutation, assuming the earth to consist of a solid spheroidal shell filled with fluid. To present the problem under its most simple form I shall first suppose the solid shell to be bounded by a determinate inner spheroidal surface, of which the ellipticity is equal to that of the outer surface, the change from the solidity of the shell to the fluidity of the included mass not being gradual but abrupt. I shall also here suppose both the shell and fluid homogeneous and of equal density. From this I propose in a future memoir to pass to the case in which the earth is considered as heterogeneous.

§. *Statement of the Problem.*

1. If S denote the position of the sun, A the centre of the earth, A P its axis of instantaneous rotation, the sun's attraction tends to produce an angular velocity of the earth about an axis through A, and perpendicular to the plane S A P. The moving force producing this rotation (supposing the earth a homogeneous spheroid),

$$= \frac{3\mu}{2r_{\prime}^3} \cdot \frac{4\pi}{15} a_{\prime}^2 c_{\prime} (a_{\prime}^2 - c_{\prime}^2) \sin 2\Delta^*,$$

where

μ = absolute force of the sun's attraction.

Δ = sun's polar distance.

$r_{\prime}$ = S A.

$a_{\prime}$ = equatorial radius.

$c_{\prime}$ = polar radius.

Also the moment of inertia of the spheroid about the axis of this rotatory motion,

$$= \frac{4\pi}{15} k\, a_{\prime}^2 c_{\prime} (a_{\prime}^2 + c_{\prime}^2).$$

Consequently the accelerating force of rotation

$$= \frac{3}{2} \cdot \frac{\mu}{r_{\prime}^3} \cdot \frac{a_{\prime}^2 - c_{\prime}^2}{a_{\prime}^2 + c_{\prime}^2} \sin 2\Delta$$

$$= \frac{3}{2} \cdot \frac{\mu}{r_{\prime}^3} \varepsilon \sin 2\Delta$$

(ε = ellipticity of the spheroid); and if we denote this quantity by α, and the diurnal

* AIRY's Tracts, Precession and Nutation.

angular velocity of the earth by ω, the angular velocity of A P about A will $= \frac{\alpha}{\omega}$, the instantaneous motion of P being perpendicular to the plane S A P*.

2. But let us now suppose the spheroid hollow, the hollow part being spherical, and having its centre coincident with that of the spheroid. The moving force of rotation will be unaltered, but the moment of inertia will

$$= \frac{4\pi}{15} k\, a_{\prime}^2 c_{\prime} (a_{\prime}^2 + c_{\prime}^2) - \frac{8\pi}{15} k\, r^5$$

(r = radius of the hollow sphere). Therefore α will now

$$= \frac{3}{2} \cdot \frac{\mu}{r_{\prime}^3} \cdot \frac{a_{\prime}^2 c_{\prime} (a_{\prime}^2 + c_{\prime}^2)}{a_{\prime}^2 c_{\prime} (a_{\prime}^2 + c_{\prime}^2) - 2 r^5} \sin 2\Delta,$$

which, if r be considerable, will be much greater than its former value.

3. Again, let us suppose this hollow sphere filled with matter in a state of perfect fluidity. The pressures of this fluid on the interior spherical surface of the shell containing it being normal pressures (whatever be the causes producing them), their directions will all pass through the centre of the spheroid, and cannot therefore influence the rotatory motion we are now considering; and since there will be no friction with the assumed perfect fluidity of the interior matter, the value of α will be precisely the same as that above stated, when the internal sphere is entirely empty. A much greater motion of the pole would therefore result from this constitution of the spheroid than if it were perfectly solid; and it would, moreover, be entirely independent of the position of the axis of rotation of the internal fluid.

4. If the internal surface of the solid shell be spheroidal instead of spherical, the directions of the fluid normal pressures will no longer pass through an axis through the centre of the earth; and when the axes of diurnal rotation of the solid shell, and of the internal fluid do not coincide (as must generally be the case from the different actions of the sun and moon on the solid shell and on the fluid contained in it), the fluid pressure arising from the centrifugal force will introduce a new and important element into the calculation of the precessional motion of the pole. I shall now proceed to the determination of this motion on the hypotheses previously stated.

§. *Formation of the Differential Equation for the Motion of the Pole.*

5. Conceive a sphere of radius unity described about the centre of the earth, which centre we shall always denote by A. Let Π (fig. 1.) be the point in which a line through the centre and perpendicular to the plane of the ecliptic meets the sphere; and P and P′ the points in which it is met respectively by the axes of instantaneous rotation of the solid shell and of the internal fluid mass†. Let P and P′ be referred to the small circle O M, of which Π is the pole, and to great circles Π P M, Π P′ M′ respect-

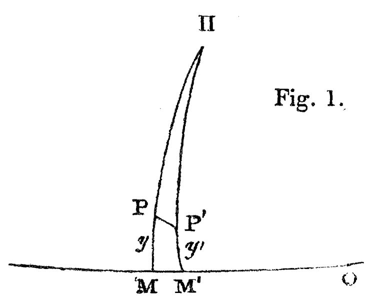

Fig. 1.

* Airy's Tracts, p. 197.

† The axis of instantaneous rotation may be regarded as coincident with the spheroidal axis of the earth;

ively, Π M being very nearly equal to the obliquity of the ecliptic. Take O an arbitrary fixed point in the small circle, and let

$$\text{O M} = x \qquad \text{O M}' = x'$$
$$\text{M P} = y \qquad \text{M}'\text{P}' = y',$$

y, y' and $x - x'$ will be in general very small, and may, therefore, be considered as straight lines. Our object will be to form a system of four simultaneous differential equations, the integration of which will give x, y, x', and y' as functions of t. For this purpose I shall first consider the *arguments* of the different terms in the expressions for $\frac{dx}{dt}, \frac{dx'}{dt}, \frac{dy}{dt}$ and $\frac{dy'}{dt}$, which severally express the effects of the different physical causes affecting the motions of P and P′, postponing the calculation of the numerical values of the *coefficients* till we shall have integrated our differential equations, as we shall then have the advantage of knowing what degree of accuracy may be essential in the determination of these values.

I. *The Attraction of the Sun on the Solid Shell.*—This will produce effects of precisely the same kind as if the spheroid were solid, but with different coefficients (Art. 2.), and therefore, if the motion of P depended on this cause alone, we should have

$$\frac{dx}{dt} = \text{A}_1 - \text{B}_1 \cos 2 (n t + \lambda)$$
$$\frac{dy}{dt} = \qquad \text{D}_1 \sin 2 (n t + \lambda),$$

(where $n t + \lambda$ is the longitude of the sun at the time t), these being the forms of the expressions which give the precessional motion of the pole, and its motion of nutation as far as they depend on the sun's action.

II. *The Attraction of the Moon on the Shell.*—This alone would give us

$$\frac{dx}{dt} = \text{A}_2 - \text{B}_2 \cos 2 (n' t + \lambda')$$
$$\frac{dy}{dt} = \qquad \text{D}_2 \sin 2 (n' t + \lambda'),$$

where $n' = \frac{\pi}{\text{period of ☽'s node}}$.

III. *The Interior Pressure on the Shell from the Attraction of the Sun on the Fluid Mass.*—If the whole mass of the earth were perfectly fluid, and its undisturbed form spherical, the attraction of the sun alone would transform this sphere into a prolate spheroid, of which the longer axis would lie in the line through S and A, the centres of the sun and earth; and similarly if the interior surface of the solid shell which we suppose to contain the internal fluid were spherical, the sun's action would tend to make this fluid assume the spheroidal form just mentioned, and would consequently produce a fluid pressure on the interior surface of the solid shell, which

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for the greatest angular separation will be of the same order as ϵ . P A P′, and may therefore be neglected in comparison with P A P′.

would be equal at all points similarly situated with respect to the line just mentioned through the centres of the sun and earth. If the interior surface of the shell be spheroidal, but of small eccentricity, very nearly the same effect will be produced. The pressure in this case will be exactly equal at points similarly situated with respect to a plane through the sun and the axis of rotation (A P) of the shell, and will consequently tend to communicate a rotatory motion to the shell about an axis perpendicular to this plane and through the earth's centre; *i.e.* about the same axis as that about which the attraction mentioned in (I.) tends to communicate a rotatory motion. Also the effects of this pressure must *recur* with recurring positions of the sun exactly in the same manner as the effects of the sun's attraction just alluded to. Hence the terms depending on this cause will be of the same form as those in (I.), as will, in fact, be proved to be the case when we come to investigate their exact value. They will, therefore, give us

$$\frac{dx}{dt} = A_3 - B_3 \cos 2(nt + \lambda)$$

$$\frac{dy}{dt} = D_3 \sin 2(nt + \lambda).$$

IV. *The Interior Pressure on the Shell from the Attraction of the Moon on the Fluid Mass.*—This will give us terms similar to those arising from the sun's action. From this cause alone, therefore, we should have

$$\frac{dx}{dt} = A_4 - B_4 \cos 2(n't + \lambda')$$

$$\frac{dy}{dt} = D_4 \sin 2(n't + \lambda').$$

V. *The Interior Pressure on the Shell from the Centrifugal Force of the Fluid Mass.*—When P and P′ do not coincide, the interior fluid mass will tend, from the effect of centrifugal force, to assume a form different from that of the interior surface of the solid shell. Thus normal pressures will be produced on the interior surface of the shell; and they will manifestly act symmetrically with respect to a plane through P, P′ and A the centre of the earth, i. e. through the axes of rotation of the solid shell and of the fluid mass. Consequently the tendency of these pressures will be to communicate a motion of rotation to the shell about an axis through A, and perpendicular to this plane; and the consequent motion of P, if this force alone acted on the shell, would be perpendicular to P′ P, the axis of rotation of the shell having, from this cause, an angular velocity in space $= \frac{\alpha''}{\omega}$ (Art. 1.) α'', being the quantity analogous to α in the article referred to; or since P and P′ are supposed to be on the surface of a sphere whose radius is unity, $\frac{\alpha''}{\omega}$ will be the *linear* velocity of P perpendicular to P′ P. Now when we come to the calculation of the quantities involved in these investigations, we shall find that $\frac{\alpha''}{\omega} = \gamma_1 \sin 2\beta$, where γ_1 is a constant quantity depending on the diurnal angular velocity (ω), and on the magnitudes and ellipticity of the fluid spheroid and solid shell; and where $\beta =$ the angle P A P′, or $=$ the line P′ P. Consequently,

$$\frac{\alpha''}{\omega} = \gamma_1 \sin 2 \,.\, \mathrm{P'\,P},$$

or, since P′ P will always be extremely small, the linear velocity of P perpendicular to P′ P,

$$= \frac{\alpha''}{\omega} = 2\,\gamma_1 \,.\, \mathrm{P'\,P};$$

and resolving this in directions parallel and perpendicular to M′ M, we have (ψ being the angle which P′ P makes with the axis of x)

$$\frac{d\,x}{d\,t} = -\,2\,\gamma_1 \,.\, \mathrm{P'\,P} \,.\, \sin\psi = -\,2\,\gamma_1\,(y - y')$$

$$\frac{d\,y}{d\,t} = 2\,\gamma_1 \,.\, \mathrm{P'\,P}\cos\psi = 2\,\gamma_1\,(x - x').$$

6. If we now take the sum of the different terms which express the effects of the several causes affecting the motion of P, we obtain for the complete values of $\frac{d\,x}{d\,t}$ and $\frac{d\,y}{d\,t}$,

$$\frac{d\,x}{d\,t} = (\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 + \mathrm{A}_4) - (\mathrm{B}_1 + \mathrm{B}_3)\cos 2\,(n\,t + \lambda) - (\mathrm{B}_2 + \mathrm{B}_4)\cos 2\,(n'\,t + \lambda') - 2\,\gamma_1\,(y - y');$$

$$\frac{d\,y}{d\,t} = (\mathrm{D}_1 + \mathrm{D}_3)\sin 2\,(n\,t + \lambda) + (\mathrm{D}_2 + \mathrm{D}_4)\sin 2\,(n'\,t + \lambda') + 2\,\gamma_1\,(x - x');$$

or putting

$$\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 + \mathrm{A}_4 = \mathrm{A}$$
$$\mathrm{B}_1 + \mathrm{B}_3 = \mathrm{B}$$
$$\mathrm{B}_2 + \mathrm{B}_4 = \mathrm{B}'$$
$$\mathrm{D}_1 + \mathrm{D}_3 = \mathrm{D}$$
$$\mathrm{D}_2 + \mathrm{D}_4 = \mathrm{D}'$$

$$\left.\begin{aligned} \frac{d\,x}{d\,t} + 2\,\gamma_1\,(y - y') &= \mathrm{A} - \mathrm{B}\cos 2\,(n\,t + \lambda) - \mathrm{B}'\cos 2\,(n'\,t + \lambda') \\ \frac{d\,y}{d\,t} - 2\,\gamma_1\,(x - x') &= \mathrm{D}\sin 2\,(n\,t + \lambda) + \mathrm{D}'\sin 2\,(n'\,t + \lambda') \end{aligned}\right\} \quad . \quad . \quad (\mathrm{A.})$$

§. *Motion of the Internal Fluid.*

7. When any accelerating forces, X, Y, Z, act upon a homogeneous fluid mass of which the whole surface or any part of it is free, we have two conditions of equilibrium, viz. that $\mathrm{X}\,d\,x + \mathrm{Y}\,d\,y + \mathrm{Z}\,d\,z$ must be a perfect derivation of a function of the three

independent variables x, y, z, and that $X\,dx + Y\,dy + Z\,dz = 0$, must be the differential equation to the free surface. If however no part of the surface of the fluid is free, as when the whole mass is contained in a rigid shell which it entirely fills, the former of these conditions is the only essential one of equilibrium. Also if there be several sets of forces which separately satisfy this condition when referred to different systems of coordinate axes, it will manifestly be satisfied by all these sets of forces taken conjointly; and if any proposed set of forces do not satisfy it, we may still omit, in the determination of the motion resulting from these forces, those terms in the expressions for X, Y, and Z, which taken conjointly do satisfy the analytical condition now spoken of. These considerations will materially simplify the following investigations.

8. We have now to consider the tendency of the forces acting on the internal fluid to put it in motion.

I. *Disturbing Force of the Sun.*—Let x, y, z be the coordinates of any particle (Q) of the internal fluid, the centre of the earth (A) being the origin, the line joining the centres of the earth and sun (A S) the axis of x, and the axis of z being perpendicular to the plane of the ecliptic. We shall then have

$$\text{the disturbing force on Q parallel to } x = 2\,\frac{\mu}{r_{\prime}^{3}} \cdot x$$

$$\text{the disturbing force on Q parallel to } y = -\,\frac{\mu}{r_{\prime}^{3}} \cdot y$$

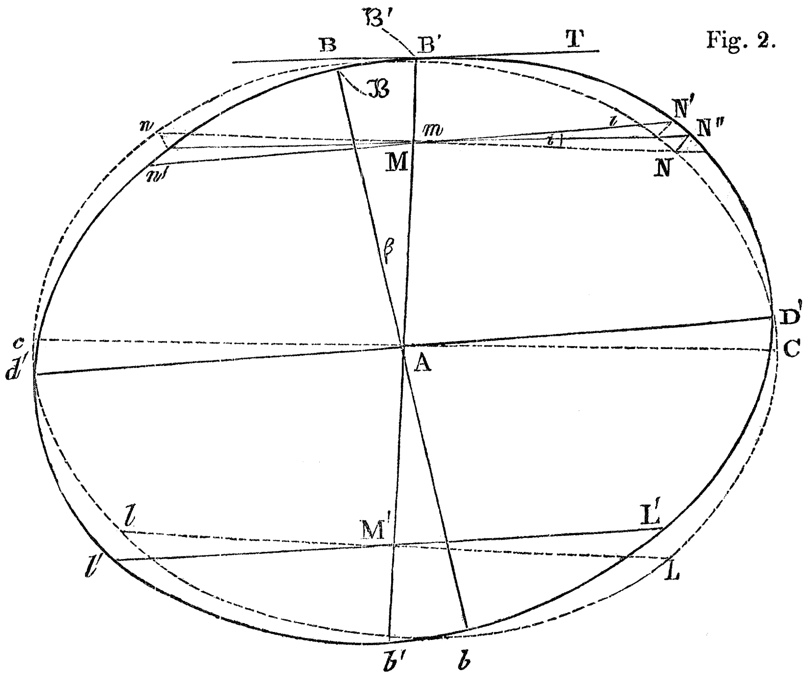
$$\text{the disturbing force on Q parallel to } z = -\,\frac{\mu}{r_{\prime}^{3}} \cdot z,$$

substituting these quantities for X, Y, and Z respectively in the expression $X\,dx + Y\,dy + Z\,dz$, it manifestly becomes a perfect derivative. Consequently the condition of equilibrium is satisfied, and the action of the sun has no tendency to communicate motion to the internal fluid.

II. *Disturbing Force of the Moon.*—The investigation and result are precisely the same as for the sun.

III. *Centrifugal Force.*—In investigating the equations of motion for the solid shell, it has been assumed (Arts. 4. 5.) that the spheroidal axis of the shell will not generally be coincident with the axis A B′ of rotation, which is now proved to be true, since the disturbing forces of the sun and moon, while they produce a motion of the shell, cause no motion by their immediate action in the fluid. Let B′ A b' (fig. 2.) be the axis of rotation of the interior fluid, and suppose the spheroidal axis first to coincide with it, the dotted ellipse then representing the section of the interior surface of the solid shell. The shell, its form being supposed coincident with that of equilibrium of the fluid, will, in this case, produce no constraint on the fluid motion; but conceive the shell to be afterwards brought into the position represented by the continuous ellipse, A B being its spheroidal axis, while B′ b' shall still represent the instantaneous axis of rotation of the fluid. It is manifest that the planes of rotatory motion of the fluid particles near B′ and b' can no longer, as in the former case, be perpendicular to A B′, but must be constrained to move in planes very nearly parallel to the tangent planes at B′ and b'; and it is also sufficiently obvious that whatever effect is produced on the

Fig. 2.



planes of motion of the above particles, a similar effect must be produced on those more remote from B′ and b'. Moreover, the mutual action of contiguous particles situated in contiguous planes of rotation will necessarily preserve a very approximate parallelism of these planes throughout the mass. We may conclude, therefore, that the instantaneous planes of rotation will always approximate, in a greater or less degree, to parallelism with the tangent planes at B′ and b', the extremities of the instantaneous axis of rotation of the fluid. In the investigations immediately following, we shall assume this to be accurately true, and shall prove subsequently the accuracy of the approximation to the true motions thus obtained. If M N′ be one of these planes of motion, M N perpendicular to A B′, and ι = angle N M N′, we shall have $\iota = 2\,\varepsilon\,\beta$, as may be easily proved.

9. The sections of the interior surface of the shell made by these planes of rotatory motion will be similar ellipses, so that the angular velocity of rotation will no longer be accurately uniform. If, however, e' be the eccentricity of these sections, ε the ellipticity of the spheroid, and β the angle B A B′ (which will always be extremely small), it is easily shown that

$$e'^2 = 2\,\varepsilon\,\beta.$$

This is so small that we shall still consider the angular velocity uniform, which will be proved in the sequel to be correct to the degree of approximation to which it is requisite to carry our investigations.

We may proceed to determine the centrifugal force on the fluid.

10. Let A B′ be now taken for the common axis of z and z', A C, perpendicular to A B′, for that of x, and A D′, conjugate to A B′, for that of x', the axis of y being perpendicular to the plane B A B′, that of the paper. x, y, z will be the coordinates of any fluid particle (Q) referred to the rectangular system of coordinates, and x', y, z' those of the same particle referred to the system in which the axis of z' is oblique to the plane of $x' y$. Also D′ A C = N M N′ = ι (Art. 8. III.). Then if r' be the distance of Q from the axis of rotation of the fluid, measured in the plane of its motion, the whole centrifugal force on Q in the direction of $r' = \omega^2 r'$, which (since x' and y are rectangular) is equivalent to $\omega^2 x'$ parallel to the axis of x', and $\omega^2 y'$ parallel to that of y. Hence

$$X = \omega^2 x' \cos \iota = \omega^2 x,$$

$$Y = \omega^2 y,$$

$$Z = \omega^2 x' \sin \iota = 2 \omega^2 \varepsilon \beta . x \text{ (Art. 8.).}$$

These forces do not satisfy the conditions of equilibrium, and therefore the assumed position will not be one of equilibrium. The conditions would be satisfied, however, if the only forces were $\omega^2 x$ and $\omega^2 y$, and consequently the only force which would tend to produce motion would be Z, or $2 \omega^2 \varepsilon \beta . x$. This is therefore the only part of the centrifugal force of which it is here necessary to take account.

11. In determining the motion produced by this force Z, we may observe, that since it acts symmetrically with respect to the plane of $x z$, by which the interior surface of the shell is divided symmetrically, there can be no motion in directions perpendicular to that plane. The motion of each fluid particle must therefore be in a plane perpendicular to the axis of y, and must moreover be independent of y, since Z is so. Hence the determination of the motion is reduced to the case of fluid motion in one plane, where (the plane itself being taken for that of $x z$) each particle is acted on by the force $Z = 2 \omega^2 \varepsilon \beta . x$, and the boundary of the fluid is an ellipse whose ellipticity is ε, and whose centre is the origin of coordinates.

12. The general equation

$$d p = X d x + Z d z,$$

where X and Z are forces which maintain the fluid in equilibrium, is easily reduced to

$$d p = R d r + \Theta r d \theta,$$

where r and θ are polar coordinates of any fluid particle, R the accelerating force upon it in the direction of r, and Θ that in a direction perpendicular to the former. Hence we have the condition of equilibrium

$$\frac{d R}{d \theta} = \frac{d . \Theta r}{d r};$$

or if Θ be the force acting on the fluid, but $\Theta + \Theta'$ that which would produce equilibrium with R, we have

$$\frac{d R}{d \theta} = \frac{d . (\Theta r + \Theta' r)}{d r}.$$

Now in the case to which this condition is to be applied, we have (θ being measured from the axis of x, and r from the origin of x and z)

$$\begin{aligned} \mathrm{R} &= \mathrm{Z} \sin \theta \\ &= 2\,\omega^2\,\varepsilon\,\beta\,r \cos \theta \sin \theta \\ &= \frac{k}{2}\,r \sin 2\,\theta, \qquad\qquad (k = 2\,\omega^2\,\varepsilon\,\beta)\,; \\ \Theta\,r &= \mathrm{Z} \cos \theta \,.\, r \\ &= k\,r^2 \cos^2 \theta, \\ \therefore \frac{d\,\mathrm{R}}{d\,\theta} &= k\,r \cos 2\,\theta, \\ \frac{d\,.\,\Theta\,r}{d\,r} &= 2\,k\,r \cos^2 \theta \\ &= k\,r\,(1 + \cos 2\,\theta). \end{aligned}$$

Substituting these values in the above equation,

$$k\,r \cos 2\,\theta = k\,r\,(1 + \cos 2\,\theta) + \frac{d\,.\,\Theta'\,r}{d\,r},$$

$$\therefore \frac{d\,.\,\Theta'\,r}{d\,r} = -\,k\,r,$$

$$\Theta'\,r = -\,\frac{k}{2}\,r^2 + \Phi\,(\theta)\,;$$

and since $\Theta'\,r$ must vanish with r, $\Phi\,(\theta)$ must $= 0$, and

$$\Theta' = -\,\frac{k}{2}\,r\,;$$

or if forces $\Theta' = -\,\frac{k}{2}\,r$, and $\mathrm{Z} = k\,x$ act on each fluid particle, there will be no motion.

Now suppose forces $\frac{k}{2}\,r$ and $-\,\frac{k}{2}\,r$ equal and in opposite directions to act on each particle perpendicular to r, together with Z. The motion produced by Z will not be affected by this superposition. But the forces Z ($= k\,x$) and $-\,\frac{k}{2}\,r$ are in equilibrium, and therefore the motion produced by Z must be the same as that which would be produced by $\frac{k}{2}\,r$, acting perpendicular to r.

13. Since the motion we are considering is in space of two dimensions, the surface of the fluid must be defined by some plane curve, if the particular form of which the result at which we have just arrived is quite independent, being subject only to the condition that no part of the fluid surface shall be free. Let us suppose the curve to be a circle, of which the centre is the origin of coordinates. The angular accelerating force on each particle $= \frac{k}{2}$, and is, therefore, the same for every particle. Also the

reaction of the surface would, in this case, have no effect on the angular motion of the fluid. Consequently the angular velocity generated by Z in a unit of time would $= \frac{k}{2}$.

14. If the boundary of the fluid be an ellipse of which the centre is the origin and the eccentricity very small, the same result will manifestly be very approximately true.

This last is the case, in which it was necessary to determine the angular motion (Art. 11.). It follows that the angular velocity of the internal fluid mass round the axis of y, which would be generated by the force Z in a unit of time $= \frac{k}{2}$, or (substituting the proper value of k) it $= \omega^2 \varepsilon \beta$, neglecting quantities of the order $\varepsilon^2 \beta$.

15. This angular velocity will be compounded with that about the axis of z (ω). Now if we again suppose the fluid mass to be spherical, it would manifestly move precisely as if it were solid, since the angular velocities ω and $\omega^2 \varepsilon \beta$ about the axes of z and y respectively are common to all the particles of the mass, and the axis of instantaneous rotation would consequently have an angular motion in space perpendicular to the plane B A B′ (fig. 2.) and $= \frac{\alpha'}{\omega} = \omega \varepsilon \beta$. If the fluid mass be spheroidal, as in our actual case, the ellipticity being small the same result will be very approximately true.

We may now proceed to the formation of the differential equations for the motion of the instantaneous axis of rotation of the interior fluid, or of the point P′ (fig. 1.).

§. *Formation of the Differential Equations for the Motion of* P′ (fig. 1.).

16. Since the angular velocity of A B′ (fig. 2.) is $\omega \varepsilon . \beta$ (Art. 15.) in a direction perpendicular to the plane B A B′, the linear velocity of P′ (fig. 1.) will also be $\omega \varepsilon . \beta$, or $\omega \varepsilon$. P P′; or if $\omega \varepsilon = 2 \gamma_2$, the linear velocity of P′ perpendicular to P P′ $= 2 \gamma_2$. P P′. This is exactly similar to the expression for the motion of P perpendicular to P P′ (Art. 5. V.), but it will be observed that the angular motion of the fluid about the axis perpendicular to the plane B A B′ (fig. 2.) which the centrifugal force tends to produce, is in the direction opposite to that of the angular motion of the shell which the fluid pressure on its interior surface, arising from this centrifugal force, tends to produce (Art. 14.). Hence to obtain the differential equations for the motion of P′ we have only to put $-\gamma_2$ for γ_1 in the equations of Art. 5. V. We thus have (now denoting by $x\ y\ x'$ and y' the same quantities as in Art. 5.).

$$\left.\begin{aligned} \frac{d x'}{d t} - 2 \gamma_2 (y - y') = 0 \\ \frac{d y'}{d t} + 2 \gamma_2 (x - x') = 0 \end{aligned}\right\} \quad \text{. . . . . (B.)}$$

§. *Integration of the Differential Equations for the Motions of* P *and* P′.

17. The equations (A) and (B) (Arts. 6, 16.) form a system of four simultaneous differential equations, viz.

$$\frac{dx}{dt} + 2\gamma_1(y - y') = A - B\cos 2(nt + \lambda) - B'\cos 2(n't + \lambda'), \quad . \quad . \quad . \quad (1.)$$

$$\frac{dy}{dt} - 2\gamma_1(x - x') = \qquad D\sin 2(nt + \lambda) + D'\sin 2(n't + \lambda'), \quad . \quad . \quad . \quad (2.)$$

$$\frac{dx'}{dt} - 2\gamma_2(y - y') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$\frac{dy'}{dt} + 2\gamma_2(x - x') = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

(1.) $\times \gamma_2$ + (3.) $\times \gamma_1$ gives

$$\gamma_2\frac{dx}{dt} + \gamma_1\frac{dx'}{dt} = \gamma_2 A - \gamma_2 B\cos 2(nt + \lambda) - \gamma_2 B'\cos 2(n't + \lambda'),$$

and (2.) $\times \gamma_2$ + (4.) $\times \gamma_1$ gives

$$\gamma_2\frac{dy}{dt} + \gamma_1\frac{dy'}{dt} = \gamma_2 D\sin 2(nt + \lambda) + \gamma_2 D'\sin 2(n't + \lambda').$$

Integrating these two last equations,

$$\gamma_2 x + \gamma_1 x' = \gamma_2 A t - \frac{\gamma_2 B}{2n}\sin 2(nt + \lambda) - \frac{\gamma_2 B'}{2n'}\sin 2(n't + \lambda') + c_1, \quad . \quad . \quad (5.)$$

$$\gamma_2 y + \gamma_1 y' = -\frac{\gamma_2 D}{2n}\cos 2(nt + \lambda) - \frac{\gamma_2 D}{2n'}\cos 2(n't + \lambda') + c_2. \quad . \quad . \quad . \quad (6.)$$

To determine the arbitrary constants c_1 and c_2, let x, y, x' and y' each $= 0$ when $t = 0$. Then

$$c_1 = \frac{\gamma_2 B}{2n}\sin 2\lambda + \frac{\gamma_2 B'}{2n'}\sin 2\lambda',$$

$$c_2 = \frac{\gamma_2 D}{2n}\cos 2\lambda + \frac{\gamma_2 D'}{2n'}\cos 2\lambda'.$$

Equations (5.) and (6.) are two integrals of our four differential equations.

18. Substituting in (1.) and (2.) for y' and x', we obtain

$$\frac{dx}{dt} + 2(\gamma_1 + \gamma_2)y = 2c_2 + A - \left(B + \frac{\gamma_2 D}{n}\right)\cos 2(nt + \lambda)$$
$$- \left(B' + \frac{\gamma_2 D'}{n'}\right)\cos 2(n't + \lambda'),$$

$$\frac{dy}{dt} - 2(\gamma_1 + \gamma_2)x = -2c_1 + \left(D + \frac{\gamma_2 B}{n}\right)\sin 2(nt + \lambda)$$
$$+ \left(D' + \frac{\gamma_2 B'}{n'}\right)\sin 2(n't + \lambda') - 2\gamma_2 A t.$$

Let $\qquad \gamma = \gamma_1 + \gamma_2, \qquad K = \gamma_2 A,$

$\qquad L = 2c_2 + A, \qquad L' = -2c_1,$

$$M = B + \frac{\gamma_2 D}{n}, \qquad M' = B' + \frac{\gamma_2 D'}{n'},$$

$$N = D + \frac{\gamma_2 B}{n}, \qquad N' = D' + \frac{\gamma_2 B'}{n'}.$$

Then

$$\frac{dx}{dt} + 2\gamma y = L - M\cos 2(nt+\lambda) - M'\cos 2(n't+\lambda'),$$

$$\frac{dy}{dt} - 2\gamma x = L' + N\sin 2(nt+\lambda) + N'\sin 2(n't+\lambda') - 2Kt.$$

The integration of these equations will be easily effected by the method of the variation of the arbitrary constants. The simultaneous equations

$$\frac{dx}{dt} + 2\gamma y = 0,$$

$$\frac{dy}{dt} - 2\gamma x = 0$$

are satisfied by

$$\left.\begin{array}{l} x = C_1\cos 2\gamma t - C_2\sin 2\gamma t \\ y = C_1\sin 2\gamma t + C_2\cos 2\gamma t \end{array}\right\} \quad \ldots\ldots \quad (C.)$$

and if we write equations (B.) under the form

$$\frac{dx}{dt} + 2\gamma y = \Phi(t),$$

$$\frac{dx}{dt} - 2\gamma x = \Psi(t),$$

we shall have, considering C_1 and C_2 as functions of t,

$$\frac{dC_1}{dt}\cos 2\gamma t - \frac{dC_2}{dt}\sin 2\gamma t = \Phi(t),$$

$$\frac{dC_1}{dt}\sin 2\gamma t + \frac{dC_2}{dt}\cos 2\gamma t = \Psi(t);$$

$$\therefore \frac{dC_1}{dt} = \cos 2\gamma t\,.\,\Phi(t) + \sin 2\gamma t\,.\,\Psi(t),$$

$$\frac{dC_2}{dt} = -\sin 2\gamma t\,.\,\Phi(t) + \cos 2\gamma t\,\Psi(t).$$

Each term in C_1 and C_2 corresponding to the several terms in $\Phi(t)$ and $\Psi(t)$ may be determined separately.

Let $\Phi(t) = L$, $\Psi(t) = L'$; then

$$\frac{dC_1}{dt} = L\,.\cos 2\gamma t + L'\sin 2\gamma t,$$

$$\frac{dC_2}{dt} = -L\sin 2\gamma t + L'\cos 2\gamma t;$$

$$\therefore C_1 = \frac{L}{2\gamma} \sin 2\gamma t - \frac{L'}{2\gamma} \cos 2\gamma t,$$

$$C_2 = \frac{L}{2\gamma} \cos 2\gamma t + \frac{L'}{2\gamma} \sin 2\gamma t.$$

Let $\Phi(t) = - M \cos 2(nt + \lambda)$,

$\Psi(t) = \quad N \sin 2(nt + \lambda)$; then

$$\frac{dC_1}{dt} = - M \cos 2\gamma t \,.\, \cos 2(nt + \lambda) + N \sin 2\gamma t \,.\, \sin 2(nt + \lambda),$$

$$\frac{dC_2}{dt} = \quad M \sin 2\gamma t \,.\, \cos 2(nt + \lambda) + N \cos 2\gamma t \,.\, \sin 2(nt + \lambda);$$

$$\therefore \frac{dC_1}{dt} = - \frac{M}{2} \left\{ \cos 2(\overline{\gamma - n}\,t - \lambda) + \cos 2(\overline{\gamma + n}\,t + \lambda) \right\}$$

$$+ \frac{N}{2} \left\{ \cos 2(\overline{\gamma - n}\,t - \lambda) - \cos 2(\overline{\gamma + n}\,t + \lambda) \right\};$$

and $\therefore C_1 = - \frac{M - N}{2} \cdot \frac{\sin 2(\overline{\gamma - n}\,t - \lambda)}{2(\gamma - n)} - \frac{M + N}{2} \frac{\sin 2(\overline{\gamma + n}\,t + \lambda)}{2(\gamma + n)}:$

$$\frac{dC_2}{dt} = \frac{M}{2} \left\{ \sin 2(\overline{\gamma + n}\,t + \lambda) + \sin 2(\overline{\gamma - n}\,t - \lambda) \right\}$$

$$+ \frac{N}{2} \left\{ \sin 2(\overline{\gamma + n}\,t + \lambda) - \sin 2(\overline{\gamma - n}\,t - \lambda) \right\};$$

and $\therefore C_2 = - \frac{M + N}{2} \cdot \frac{\cos 2(\overline{\gamma + n}\,t + \lambda)}{2(\gamma + n)} - \frac{M - N}{2} \cdot \frac{\cos 2(\overline{\gamma - n}\,t - \lambda)}{2(\gamma - n)}.$

Taking $\varphi(t) = - M' \cos 2(n't + \lambda')$,

$\psi(t) = \quad N' \sin 2(n't + \lambda')$,

we shall obtain in a similar manner

$$C_1 = - \frac{M' - N'}{2} \cdot \frac{\sin 2(\overline{\gamma - n'}\,t - \lambda')}{2(\gamma - n')} - \frac{M' + N'}{2} \cdot \frac{\sin 2(\overline{\gamma + n'}\,t + \lambda')}{2(\gamma + n')},$$

$$C_2 = - \frac{M' + N'}{2} \cdot \frac{\cos 2(\overline{\gamma + n'}\,t + \lambda')}{2(\gamma + n')} - \frac{M' - N'}{2} \cdot \frac{\cos 2(\overline{\gamma - n'}\,t - \lambda')}{2(\gamma - n')}.$$

Let $\varphi(t) = 0$, $\psi(t) = - 2Kt$; then

$$\frac{dC_1}{dt} = - 2Kt \,.\, \sin 2\gamma t,$$

$$\frac{dC_2}{dt} = - 2Kt \,.\, \cos 2\gamma t;$$

$$\therefore C_1 = - 2K \left(\frac{\sin 2\gamma t}{4\gamma^2} - \frac{\cos 2\gamma t}{2\gamma} \,.\, t \right),$$

$$C_2 = - 2K \left(\frac{\cos 2\gamma t}{4\gamma^2} + \frac{\sin 2\gamma t}{2\gamma} \cdot t \right).$$

Hence for the complete values of C_1 and C_2 we have

$$C_1 = \frac{L}{2\gamma}\sin 2\gamma t - \frac{L'}{2\gamma}\cos 2\gamma t$$
$$- \frac{M-N}{4(\gamma-n)}\sin 2(\overline{\gamma-n}\,t-\lambda) - \frac{M+N}{4(\gamma+n)}\sin 2(\overline{\gamma+n}\,t+\lambda)$$
$$- \frac{M'-N'}{4(\gamma-n')}\sin 2(\overline{\gamma-n'}\,t-\lambda') - \frac{M'+N'}{4(\gamma+n')}\sin 2(\overline{\gamma+n'}\,t+\lambda')$$
$$- 2K.\left(\frac{\sin 2\gamma t}{4\gamma^2} - \frac{\cos 2\gamma t}{2\gamma}t\right) + c_3;$$

$$C_2 = \frac{L}{2\gamma}\cos 2\gamma t + \frac{L'}{2\gamma}\sin 2\gamma t$$
$$- \frac{M+N}{4(\gamma+n)}\cos 2(\overline{\gamma+n}\,t+\lambda) - \frac{M-N}{4(\gamma-n)}\cos 2(\overline{\gamma-n}\,t-\lambda)$$
$$- \frac{M'+N'}{4(\gamma+n')}\cos 2(\overline{\gamma+n'}\,t+\lambda') - \frac{M'-N'}{4(\gamma-n')}\cos 2(\overline{\gamma-n'}\,t-\lambda')$$
$$- 2K\left(\frac{\cos 2\gamma t}{4\gamma^2} + \frac{\sin 2\gamma t}{2\gamma}\right) + c_4,$$

where c_3 and c_4 are arbitrary constants.

Substituting these values in equations (C.), we obtain after reduction,

$$\left.\begin{array}{l} x = -\frac{L'}{2\gamma} + \frac{nM-\gamma N}{2(\gamma^2-n^2)}\sin 2(nt+\lambda) + \frac{n'M'-\gamma N'}{2(\gamma^2-n'^2)}\sin 2(n't+\lambda') \\ \quad + c_3\cos 2\gamma t - c_4 \sin 2\gamma t + \frac{K}{\gamma}\cdot t; \end{array}\right\} \quad . \quad . \quad (7.)$$

$$\left.\begin{array}{l} y = \frac{L}{2\gamma} - \frac{K}{2\gamma^2} - \frac{\gamma M - nN}{2(\gamma^2-n^2)}\cos 2(nt+\lambda) - \frac{\gamma M' - n'N'}{2(\gamma^2-n'^2)}\cos 2(n't+\lambda') \\ \quad + c_3 \sin 2\gamma t + c_4\cos 2\gamma t. \end{array}\right\} \quad (8.)$$

(7.) and (8.) together: (D.)

To determine the arbitrary constants c_3 and c_4, we have $x = 0$ and $y = 0$ when $t = 0$, which gives

$$c_3 = -\frac{nM-\gamma N}{2(\gamma^2-n^2)}\sin 2\lambda - \frac{n'M'-\gamma N'}{2(\gamma^2-n'^2)}\sin 2\lambda' + \frac{L'}{2\gamma},$$

$$c_4 = \frac{\gamma M - nN}{2(\gamma^2-n^2)}\cos 2\lambda + \frac{\gamma M'-n'N'}{2(\gamma^2-n'^2)}\cos 2\lambda' - \left(\frac{L}{2\gamma} - \frac{K}{2\gamma^2}\right).$$

Equations (5) (6) (7) and (8) are the four integrals of our four differential equations.

We have now to express the coefficients of equations (D.) in terms of A, B, B', D, and D'.

$$-\frac{L'}{2\gamma} = \frac{\gamma_2}{\gamma}\left(\frac{B}{2n}\sin 2\lambda + \frac{B'}{2n'}\sin 2\lambda'\right);$$

$$\frac{L}{2\gamma} - \frac{K}{2\gamma^2} = \frac{\gamma_2}{\gamma}\left(\frac{D}{2n}\cos 2\lambda + \frac{D'}{2n'}\cos 2\lambda'\right) + \frac{1}{\gamma}\frac{A}{2} - \frac{\gamma_2}{\gamma^2}\frac{A}{2}$$

$$= \frac{\gamma_2}{\gamma}\left(\frac{D}{2\,n}\cos 2\,\lambda + \frac{D'}{2\,n'}\cos 2\,\lambda'\right) + \frac{\gamma_1}{\gamma^2}\cdot\frac{A}{2}.$$

$$n\,M - \gamma\,N = n\,B - \gamma_2\,D - \gamma\,D - \gamma\frac{\gamma_2}{n}B$$

$$= \left\{-\frac{\gamma^2-n^2}{n}\frac{B}{D} + \gamma_1\left(\frac{\gamma}{n}\frac{B}{D} - 1\right)\right\}D;$$

$$\therefore \frac{n\,M-\gamma\,N}{2\,(\gamma^2-n^2)} = \left\{-\frac{1}{n}\frac{B}{D} + \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n}\frac{B}{D} - 1\right)\right\}\frac{D}{2}.$$

Similarly,

$$\frac{n'\,M'-\gamma\,N'}{2\,(\gamma^2-n'^2)} = \left\{-\frac{1}{n'}\frac{B'}{D'} + \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}\frac{B'}{D'} - 1\right)\right\}\frac{D'}{2}.$$

$$\gamma\,M - n\,N = \gamma\,B + \gamma\frac{\gamma_2}{n}D - n\,D - \gamma_2\,B$$

$$= \left\{\frac{\gamma^2-n^2}{n} - \gamma_1\left(\frac{\gamma}{n} - \frac{B}{D}\right)\right\}D;$$

$$\therefore -\frac{\gamma\,M-n\,N}{2\,(\gamma^2-n^2)} = -\left\{\frac{1}{n} - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n} - \frac{B}{D}\right)\right\}\frac{D}{2}.$$

Similarly,

$$-\frac{\gamma\,M'-n'\,N'}{2\,(\gamma^2-n'^2)} = -\left\{\frac{1}{n'} - \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'} - \frac{B'}{D'}\right)\right\}\frac{D'}{2}.$$

$$\frac{K}{\gamma} = \frac{\gamma_2}{\gamma}\cdot A.$$

$$c_3 = \left\{\frac{\gamma_1}{\gamma}\frac{B}{D}\frac{1}{n} - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n}\frac{B}{D} - 1\right)\right\}\frac{D}{2}\sin 2\,\lambda$$
$$+ \left\{\frac{\gamma_1}{\gamma}\cdot\frac{B'}{D'}\frac{1}{n'} - \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}\frac{B}{D} - 1\right)\right\}\frac{D'}{2}\sin 2\,\lambda';$$

$$c_4 = \left\{\frac{\gamma_1}{\gamma}\frac{1}{n} - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n} - \frac{B}{D}\right)\right\}\frac{D}{2}\cos 2\,\lambda$$
$$+ \left\{\frac{\gamma_1}{\gamma}\frac{1}{n'} - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n'} - \frac{B'}{D'}\right)\right\}\frac{D'}{2}\cos 2\,\lambda' - \frac{\gamma_1}{\gamma^2}\frac{A}{2}.$$

Hence we obtain by substitution,

$$\left.\begin{array}{l} x = \left\{-\frac{B}{2\,n} + \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n}\frac{B}{D} - 1\right)\frac{D}{2}\right\}\sin 2\,(n\,t+\lambda) \\ \quad + \left\{-\frac{B'}{2\,n'} + \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}\frac{B'}{D'} - 1\right)\frac{D'}{2}\right\}\sin 2\,(n'\,t+\lambda') \end{array}\right\} \quad \text{(E.)}$$

$$+ \left\{\begin{array}{l} \frac{\gamma_1}{\gamma}\cdot\frac{B}{2\,n}\sin 2\,\lambda - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n}\frac{B}{D} - 1\right)\frac{D}{2}\sin 2\,\lambda \\ + \frac{\gamma_1}{\gamma}\cdot\frac{B'}{2\,n'}\sin 2\,\lambda' - \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}\frac{B'}{D'} - 1\right)\frac{D'}{2}\sin 2\,\lambda' \end{array}\right\}\cos 2\,\gamma\,t$$

$$-\left\{\begin{array}{l}\frac{\gamma_1}{\gamma}\cdot\frac{D}{2n}\cdot\cos 2\lambda - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n}-\frac{B}{D}\right)\frac{D}{2}\cos 2\lambda \\ +\frac{\gamma_1}{\gamma}\cdot\frac{D'}{2n'}\cos 2\lambda' - \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}-\frac{B'}{D'}\right)\frac{D'}{2}\cos 2\lambda' - \frac{\gamma_1}{\gamma^2}\frac{A}{2}\end{array}\right\}\sin 2\gamma t$$

$$+\frac{\gamma_2}{\gamma}A t + \frac{\gamma_2}{\gamma}\left(\frac{B}{2n}\sin 2\lambda + \frac{B'}{2n'}\sin 2\lambda'\right);$$

$$y = -\left\{\frac{D}{2n} - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n}-\frac{B}{D}\right)\frac{D}{2}\right\}\cos 2(nt+\lambda)$$

$$-\left\{\frac{D'}{2n'} - \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}-\frac{B'}{D'}\right)\frac{D'}{2}\right\}\cos 2(n't+\lambda')$$

$$+\left\{\begin{array}{l}\frac{\gamma_1}{\gamma}\frac{B}{2n}\sin 2\lambda - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n}\frac{B}{D}-1\right)\frac{D}{2}\sin 2\lambda \\ +\frac{\gamma_1}{\gamma}\frac{B'}{2n'}\sin 2\lambda' - \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}\frac{B'}{D'}-1\right)\frac{D'}{2}\sin 2\lambda'\end{array}\right\}\sin 2\gamma t \qquad \text{(F.)}$$

$$+\left\{\begin{array}{l}\frac{\gamma_1}{\gamma}\frac{D}{2n}\cos 2\lambda - \frac{\gamma_1}{\gamma^2-n^2}\left(\frac{\gamma}{n}-\frac{B}{D}\right)\frac{D}{2}\cos 2\lambda \\ +\frac{\gamma_1}{\gamma}\frac{D'}{2n'}\cos 2\lambda' - \frac{\gamma_1}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}-\frac{B'}{D'}\right)\frac{D'}{2}\cos 2\lambda' - \frac{\gamma_1}{\gamma^2}\cdot\frac{A}{2}\end{array}\right\}\cos 2\gamma t$$

$$+\frac{\gamma_2}{\gamma}\left(\frac{D}{2n}\cos 2\lambda + \frac{D'}{2n'}\cos 2\lambda'\right) + \frac{\gamma_1}{\gamma^2}\cdot\frac{A}{2}.$$

From equations (5.) and (6.) we have

$$x' = -\frac{\gamma_2}{\gamma_1}\frac{B}{2n}\sin 2(nt+\lambda) - \frac{\gamma_2}{\gamma_1}\frac{D'}{2n'}\sin 2(n't+\lambda') + \frac{\gamma_2}{\gamma_1}A t + \frac{c_1}{\gamma_1} - \frac{\gamma_2}{\gamma_1}x,$$

$$y' = -\frac{\gamma_2}{\gamma_1}\frac{D}{2n}\cos 2(nt+\lambda) - \frac{\gamma_2}{\gamma_1}\frac{D'}{2n'}\cos 2(n't+\lambda') + \frac{c_2}{\gamma_1} - \frac{\gamma_2}{\gamma_1}y;$$

and substituting in these expressions the above values of x and y, we have

$$x' = -\frac{\gamma_2}{\gamma^2-n^2}\left(\frac{\gamma}{n}\frac{B}{D}-1\right)\frac{D}{2}\sin 2(nt+\lambda)$$

$$-\frac{\gamma_2}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}\frac{B'}{D'}-1\right)\frac{D'}{2}\sin 2(n't+\lambda')$$

$$-\left\{\begin{array}{l}\frac{\gamma_2}{\gamma}\frac{B}{2n}\sin 2\lambda - \frac{\gamma_2}{\gamma^2-n^2}\left(\frac{\gamma}{n}\frac{B}{D}-1\right)\frac{D}{2}\sin 2\lambda \\ +\frac{\gamma_2}{\gamma}\frac{B'}{2n'}\sin 2\lambda' - \frac{\gamma_2}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}\cdot\frac{B'}{D'}-1\right)\frac{D'}{2}\sin 2\lambda'\end{array}\right\}\cos 2\gamma t \qquad \text{(G.)}$$

$$+\left\{\begin{array}{l}\frac{\gamma_2}{\gamma}\frac{D}{2n}\cos 2\lambda - \frac{\gamma_2}{\gamma^2-n^2}\left(\frac{\gamma}{n}-\frac{B}{D}\right)\frac{D}{2}\cos 2\lambda \\ +\frac{\gamma^2}{\gamma}\frac{D'}{2n'}\cos 2\lambda' - \frac{\gamma_2}{\gamma^2-n'^2}\left(\frac{\gamma}{n'}-\frac{B'}{D'}\right)\frac{D'}{2}\cos 2\lambda' - \frac{\gamma_2}{\gamma^2}\frac{A}{2}\end{array}\right\}\sin 2\gamma t$$

$$+\frac{\gamma_2}{\gamma}A t + \frac{\gamma_2}{\gamma}\left(\frac{B}{2n}\sin 2\lambda + \frac{B'}{2n'}\sin 2\lambda'\right);$$

$$
\begin{aligned}
y' = & -\frac{\gamma_2}{\gamma^2 - n^2}\left(\frac{\gamma}{n} - \frac{B}{D}\right)\frac{D}{2}\cos 2(n t + \lambda) \\
& -\frac{\gamma_2}{\gamma^2 - n'^2}\left(\frac{\gamma}{n'} - \frac{B'}{D'}\right)\frac{D'}{2}\cos 2(n' t + \lambda') \\
& -\left\{\begin{array}{l} \frac{\gamma_2}{\gamma}\frac{B}{2n}\sin 2\lambda - \frac{\gamma_2}{\gamma^2 - n^2}\left(\frac{\gamma}{n}\frac{B}{D} - 1\right)\frac{D}{2}\sin 2\lambda \\ + \frac{\gamma_2}{\gamma}\frac{B'}{2n'}\sin 2\lambda' - \frac{\gamma_2}{\gamma^2 - n'^2}\left(\frac{\gamma}{n'}\frac{B'}{D'} - 1\right)\frac{D'}{2}\sin 2\lambda' \end{array}\right\}\sin 2\gamma t \qquad \text{(H.)} \\
& -\left\{\begin{array}{l} \frac{\gamma_2}{\gamma}\frac{D}{2n}\cos 2\lambda - \frac{\gamma_2}{\gamma^2 - n^2}\left(\frac{\gamma}{n} - \frac{B}{D}\right)\frac{D}{2}\cos 2\lambda \\ + \frac{\gamma_2}{\gamma}\frac{D'}{2n'}\cos 2\lambda' - \frac{\gamma_2}{\gamma^2 - n'^2}\left(\frac{\gamma}{n'} - \frac{B'}{D'}\right)\frac{D'}{2}\cos 2\lambda' - \frac{\gamma_2}{\gamma^2}\frac{A}{2} \end{array}\right\}\cos 2\gamma t \\
& + \frac{\gamma_2}{\gamma}\left(\frac{D}{2n}\cos 2\lambda + \frac{D'}{2n'}\cos 2\lambda'\right) - \frac{\gamma_2}{\gamma^2}\frac{A}{2}.
\end{aligned}
$$

We have now to determine the values of A, B, D, B', D', γ_1 and γ_2, for which purpose (Art. 6.) we must find the values of the following quantities:

$$
\begin{array}{c}
A_1\ B_1\ D_1 \\
A_2\ B_2\ D_2 \\
A_3\ B_3\ D_3 \\
A_4\ B_4\ D_4 \\
\gamma_1\ \gamma_2.
\end{array}
$$

§. *Determination of the Numerical Values of the Constants in equations* (E), (F), (G) *and* (H).

19. *Values of* A_1, B_1, *and* D_1.—The moment of the disturbing force of the sun communicating a rotatory motion to the earth considered as a homogeneous spheroid,

$$= \frac{3\mu}{2r_1^3}\cdot\frac{4\pi}{15}a_1^2 c_1 (a_1^2 - c_1^2)\sin 2\Delta \qquad \text{(Art. 1.)}$$

$$= \frac{3\mu}{r_1^3}\cdot\frac{4\pi}{15}a_1^5\,\varepsilon\sin 2\Delta;$$

and the moment of the forces on the shell

$$= \frac{3\mu}{r_1^3}\cdot\frac{4\pi}{15}\varepsilon(a_1^5 - a^5)\sin 2\Delta$$

$$= \frac{3\mu}{r_1^3}\cdot\frac{4\pi}{15}\varepsilon a^5 (q^5 - 1)\sin 2\Delta,$$

where $q = \frac{a_1}{a}$, the ratio of the outer to the inner radius of the shell.

The moment of inertia of the shell

$$= \frac{8\pi}{15}a^5 (q^5 - 1),$$

and therefore
$$\frac{\alpha}{\omega} = \frac{3}{2}\,\frac{\mu}{r_1^3\,\omega}\,\varepsilon \sin 2\,\Delta$$
$$= \frac{6\,\pi^2}{T^2\omega}\,\varepsilon \sin 2\,\Delta \qquad (T = \text{one year})$$
$$= \frac{3\,\pi}{\nu\,T}\,\varepsilon \sin 2\,\Delta$$

(since $T\,\omega = 2\,\pi\,\nu$, $\nu = 366{\cdot}26$), which is the same as if the spheroid were entirely solid. This gives us*

$$A_1 = \frac{3\,\pi}{\nu}\,\varepsilon \sin I \cos I\,\frac{1}{T},$$
$$B_1 = \frac{3\,\pi}{\nu}\,\varepsilon \sin I \cos I\,\frac{1}{T},$$
$$D_1 = \frac{3\,\pi}{\nu}\,\varepsilon \sin I\,\frac{1}{T},$$

where I = inclination of the ecliptic.

20. *Values of* A_2, B_2, *and* D_2.—In a similar manner we obtain

$$A_2 = \frac{3\,\pi}{\nu'(\sigma + 1)}\,\varepsilon \sin I \cos I \cos^2 i\,\frac{1}{T'},$$
$$B_2 = \frac{3\,\pi}{2\,\nu'(\sigma + 1)}\,\varepsilon \cos 2\,I \sin 2\,I\,\frac{1}{T'},$$
$$D_2 = -\frac{3\,\pi}{2\,\nu'(\sigma + 1)}\,\varepsilon \cos I \sin 2\,i\,\frac{1}{T'},$$

where i = inclination of the plane of the moon's orbit to that of the ecliptic; T' = moon's sidereal period, ν' = number of days in it = 27·32; and

$$\sigma = \frac{\text{mass of the moon}}{\text{mass of the earth}} = 70 \text{ nearly}.$$

21. *Numerical value of* A_3 B_3 and D_3.—Let the interior surface of the shell be referred to three rectangular co-ordinates $x\,y\,z$, the spheroidal axis being now that of z; and let p denote the normal pressure at the point $x\,y\,z$; ξ and ζ the angles which the normal makes with lines parallel to the axes of x and z respectively. Then if

$$X = p\,.\,\delta S \cos \xi, \qquad Z = p\,.\,\delta S \cos \zeta,$$

the moment of the normal pressures with respect to the axis of y

$$= \Sigma\,(Z\,x - X\,z)$$
$$= \Sigma\,(x \cos \zeta - z \cos \xi)\,p\,.\,\delta S$$
$$= \Sigma\left(x - z\,\frac{\cos \xi}{\cos \zeta}\right) p\,\delta S \cos \zeta.$$

But
$$\cos \zeta = \frac{1}{\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}},$$

* AIRY's Tracts, p. 210.

$$\cos \xi = \frac{-\frac{dz}{dx}}{\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}}.$$

And in the spheroid

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

$$\therefore - \frac{dz}{dx} = \frac{c^2}{a^2} \cdot \frac{x}{z};$$

whence by substitution, the moment about the axis of y

$$= \Sigma \left(1 - \frac{c^2}{a^2}\right) x p \,.\, \delta \mathrm{S} \cos \zeta$$

$$= 2 \varepsilon \Sigma x p \,.\, \delta \mathrm{S} \cos \zeta,$$

where

$$\varepsilon = \frac{a - c}{a}.$$

In the determination of p it will suffice to consider the spheroidal surface as spherical; and since the disturbing force of the sun is the only force producing the rotation we are now considering, the other forces may be here neglected. Hence if the line joining the centres of the sun and earth be taken for the axis of x', and the plane through this line and the spheroidal axis for that of $x' z'$,

$$\mathrm{X} = \frac{\mu}{r_1^3} \cdot 2 x',$$

$$\mathrm{Y} = - \frac{\mu}{r_1^3} \cdot y',$$

$$\mathrm{Z} = - \frac{\mu}{r_1^3} \cdot z';$$

and

$$p = \frac{\mu}{r_1^3} \left\{ 2 x' \delta x' - y' \delta y' - z' \delta z' \right\};$$

$$\therefore p = \frac{\mu}{r_1^3} \left\{ x'^2 - \frac{1}{2} (y'^2 + z'^2) + \mathrm{C} \right\}.$$

But considering the spheroid as approximately a sphere, and $x' y' z'$ a point on its surface,

$$y'^2 + z'^2 = a^2 - x'^2,$$

$$\therefore p = \frac{1}{2} \cdot \frac{\mu}{r_1^3} (3 x'^2 - a^2 + \mathrm{C});$$

and if the plane of $x z$ coincide with that of $x' z'$

$$x' = x \sin \Delta + z \cos \Delta,$$

Δ being the sun's north polar distance, and the spheroidal axis that of z.

$$\therefore p = \frac{3}{2} \cdot \frac{\mu}{r_1^3} \left\{ x^2 \sin^2 \Delta + x z \sin 2 \Delta + z^2 \cos^2 \Delta - \frac{a^2 - \mathrm{C}}{3} \right\};$$

and the moment about the axis of y

$$= 3 \frac{\mu}{r_1^3} \varepsilon \iint \left\{ x^3 \sin^2 \Delta + x^2 z \sin 2 \Delta + z^2 x \cos^2 \Delta - \frac{a^2 - C}{3} x \right\} \delta S \cos \zeta.$$

Let

$$z = r \cos \theta,$$
$$y = r \sin \theta \sin \phi,$$
$$x = r \sin \theta \cos \phi,$$

θ being the angle which r makes with the axis of z, and ϕ that which the plane of θ makes with that of $x\,z$. Considering the spheroid to be approximately a sphere we may put $r = a$, and $\zeta = \theta$. Also we shall have

$$\delta S = a^2 \sin \theta \,.\, \delta \theta \delta \phi.$$

Hence

$$\iint x^3 \cos \zeta \,.\, \delta S = a^5 \iint \sin^4 \theta \cos \theta \cos^3 \phi \, d\theta \, d\phi,$$
$$\iint z^2 x \cos \zeta \,.\, \delta S = a^5 \iint \sin^2 \theta \cos^3 \theta \cos \phi \, d\theta \, d\phi,$$
$$\iint x \cos \zeta \,.\, \delta S = a^3 \iint \sin^2 \theta \cos \theta \cos \phi \, d\theta \, d\phi,$$
$$\iint x^2 z \cos \zeta \,.\, \delta S = a^5 \iint \sin^2 \theta \cos^2 \theta \cos^2 \phi \, d\theta \, d\phi.$$

Since $\int_0^{2\pi} \cos^{2n+1} \phi \,.\, d\phi = 0$, each of these integrals except the last will vanish between the limits $\phi = 0$ and $\phi = 2\pi$. Consequently the moment about the axis of y

$$= 3 \frac{\mu}{r_1^3} \varepsilon \sin 2 \Delta \, a^5 \iint \sin^3 \theta \cos^2 \theta \cos^2 \phi \, d\theta \, d\phi \quad \left\{ \begin{matrix} \phi = 0 \text{ to } \phi = 2\pi \\ \theta = 0 \text{ to } \theta = \pi \end{matrix} \right\}$$

$$= \frac{4\pi}{5} \frac{\mu}{r_1^3} \varepsilon \sin 2 \Delta \, a^5.$$

The moment of inertia of the shell $= \frac{8\pi}{15} a^5 (q^5 - 1)$. Consequently the accelerating force of rotation arising from the pressure we are now considering

$$= \frac{3}{2} \frac{\mu}{r_1^3} \frac{\varepsilon}{q^5 - 1} \sin 2 \Delta;$$

or if α' be the angular velocity generated by this force in a unit of time

$$\frac{\alpha'}{\omega} = \frac{3}{2} \cdot \frac{\mu}{r_1^3 \omega} \varepsilon \frac{1}{q^5 - 1} \sin 2 \Delta,$$

$$= \frac{1}{q^5 - 1} \cdot \frac{\alpha}{\omega}. \qquad \text{(Art. 19.)}$$

Hence we have

$$A_3 = \frac{A_1}{q^5 - 1},$$

$$B_3 = \frac{B_1}{q^5 - 1},$$

$$D_3 = \frac{D_1}{q^5 - 1}.$$

22. *Values of* A_4, B_4, *and* D_4.—In the same manner we obtain

$$A_4 = \frac{A_2}{q^5 - 1},$$

$$B_4 = \frac{B_2}{q^5 - 1},$$

$$D_4 = \frac{D_2}{q^5 - 1}.$$

23. *Value of* γ_1.—We have seen (Art. 5. V.) that the angular motion of the solid shell produced by the centrifugal force of the fluid will be about an axis through the centre of the spheroid, and perpendicular to the plane passing through the spheroidal axis of the shell and the axis of rotation of the fluid. Let this axis be now taken for that of y', and the axis of rotation of the fluid for that of z', and let x' y' z' be the co-ordinate of any particle of the fluid; then p denoting the fluid pressure there,

$$d\,p = \left(X' - \frac{d^2 x'}{d\,t^2}\right) d\,x' + \left(Y' - \frac{d^2 y'}{d\,t^2}\right) d\,y' + \left(Z' - \frac{d^2 z'}{d\,t^2}\right) d\,z'.$$

Now the impressed forces with which we are here concerned being those only which arise from centrifugal force on the fluid, we have (referring to Art. 12, and observing that the letters which are there unaccented are accented in our present notation)

$$X' = \omega^2 x', \quad Y' = \omega^2 y', \quad Z' = 2\,\omega^2\,\varepsilon\,\beta\,.\,x'.$$

Also, since the motion of the fluid about the axis of y' is that produced by the accelerating force $\omega^2\,\varepsilon\,\beta\,.\,r$ acting perpendicularly to r (Art. 12.), we have

$$\frac{d^2 x'}{d\,t^2} = -\,\omega^2\,\varepsilon\,\beta\,.\,z', \quad \frac{d^2 y'}{d\,t^2} = 0, \quad \frac{d^2 z'}{d\,t^2} = \omega^2\,\varepsilon\,\beta\,.\,x'.$$

Hence we have by substitution,

$$d\,p = \omega^2\,(x'\,d\,x' + y'\,d\,y') + \omega^2\,\varepsilon\,\beta\,(z'\,d\,x' + x'\,d\,z'),$$

$$\therefore p = \frac{\omega^2}{2}\,(x'^2 + y'^2) + 2\,\varepsilon\,\beta\,x'\,z' + C.$$

The moment of this force about the axis of y'

$$= 2\,\varepsilon\,\Sigma\,x\,p\,.\,\delta\,S\cos\zeta, \qquad \text{(Art. 21.)}$$

$$= \omega^2\,\varepsilon\,\Sigma\,x\,(x'^2 + y'^2 + 2\,\varepsilon\,\beta\,x'\,z' + C)\,\delta\,S\cos\zeta.$$

In this expression we may consider x' y' z' as co-ordinates of a point in the surface of a sphere, whose radius $= a$. Therefore

$$x'^2 + y'^2 = a^2 - z'^2 \text{ approximately.}$$

Also, since in our results we shall only retain terms of the order $\varepsilon\,\beta$, the term $2\,\varepsilon\,\beta\,x'\,z'$ may be neglected, the whole quantity under the integral sign being multiplied by ε. This is the term arising from the effective force on the fluid.

Hence, the moment about the axis of y

$$= \omega^2\,\varepsilon\,\Sigma\,x\,(a^2 + C - z'^2)\,\delta\,S\cos\zeta.$$

The spheroidal axis being the axis of z we have

$$z' = z \cos \beta - x \sin \beta,$$

$$z'^2 = z^2 \cos^2 \beta - z\, x \sin 2 \beta + x^2 \sin^2 \beta;$$

and substituting the expressions for x and z in terms of the polar co-ordinates as in article 21, the above expression reduces itself as in that article to

$$\omega^2 \varepsilon \sin 2 \beta \, \Sigma \, z \, x^2 \, \delta \, \text{S} \cos \zeta$$

$$= \omega^2 \varepsilon \sin 2 \beta \, a^5 \iint \sin^3 \theta \cos^2 \theta \cos^2 \varphi \, d \theta \, d \varphi$$

$$= \frac{4 \pi}{15} \omega^2 \varepsilon \sin 2 \beta \, a^5.$$

Dividing this by the moment of inertia $\left(= \frac{8 \pi}{15} a^5 (q^5 - 1)\right)$ we have (if α'' be the angular velocity generated in a unit of time by the force we are now considering)

$$\alpha'' = \frac{\omega^2 \varepsilon}{2 (q^5 - 1)} \sin 2 \beta,$$

and

$$\frac{\alpha''}{\omega} = \frac{\omega \varepsilon}{2 (q^5 - 1)} \sin 2 \beta;$$

and therefore (Art. 5, V.)

$$\gamma_1 = \frac{\omega \varepsilon}{2 (q^5 - 1)}$$

$$= \frac{\pi \varepsilon}{q^5 - 1} \frac{1}{t_1},$$

since if t_1 = one day, $\omega = \frac{2 \pi}{t_1}$.

24. *Value of* γ_2.—We have seen (Art. 16.) that

$$\gamma_2 = \frac{\omega \varepsilon}{2}$$

$$= \pi \varepsilon \, . \frac{1}{t_1}.$$

25. Substituting the values of A, B, B′, D, and D′ (Art. 6.) we obtain

$$\text{A} = \frac{q^5}{q^5 - 1} \cdot \frac{3 \pi}{\nu} \varepsilon \sin \text{I} \cos \text{I} \left(\frac{1}{\text{T}} + \frac{1}{\sigma + 1} \frac{\nu}{\nu'} \cos^2 i \frac{1}{\text{T}'}\right),$$

$$\text{B} = \frac{q^5}{q^5 - 1} \cdot \frac{3 \pi}{\nu} \varepsilon \sin \text{I} \cos \text{I} \frac{1}{\text{T}},$$

$$\text{B}' = \frac{q^5}{q^5 - 1} \cdot \frac{3 \pi}{2 \nu' (\sigma + 1)} \varepsilon \cos 2 \text{I} \sin 2 i \frac{1}{\text{T}'},$$

$$\text{D} = \frac{q^5}{q^5 - 1} \cdot \frac{3 \pi}{\nu} \varepsilon \sin \text{I} \frac{1}{\text{T}},$$

$$\text{D}' = - \frac{q^5}{q^5 - 1} \cdot \frac{3 \pi}{2 \nu' (\sigma + 1)} \varepsilon \cos \text{I} \sin 2 i \frac{1}{\text{T}'}.$$

These give us

$$\frac{B}{D} = \cos I,$$

$$\frac{B'}{D'} = -\frac{\cos 2I}{\cos I},$$

$$n = \frac{2\pi}{T}, \qquad n' = \frac{\pi}{\tau}, \qquad \text{(Art. 5, I. and II.)}$$

$$\gamma_1 = \frac{\pi \varepsilon}{q^5 - 1}\frac{1}{t_1}, \qquad \text{(Art. 23.)}$$

$$\gamma_2 = \pi \varepsilon \frac{1}{t_1}, \qquad \text{(Art. 24.)}$$

$$\therefore \gamma = \gamma_1 + \gamma_2 = \frac{q^5}{q^5 - 1}\pi \varepsilon \frac{1}{t_1}.$$

For the convenience of reference we may also put down the following ratios:

$$\frac{n}{\gamma} = 2\frac{q^5 - 1}{q^5}\cdot\frac{t_1}{\varepsilon T} = 2\frac{q^5 - 1}{q^5}\cdot\frac{1}{\nu \varepsilon}, \qquad \left(\text{since } \frac{T}{t_1} = \nu\right)$$

$$\frac{n'}{\gamma} = \frac{q^5 - 1}{q^5}\cdot\frac{t_1}{\varepsilon} = \frac{q^5 - 1}{q^5}\cdot\frac{T}{\tau}\cdot\frac{1}{\nu \varepsilon},$$

$$\frac{\gamma_1}{\gamma} = \frac{1}{q^5}, \qquad \frac{\gamma_2}{\gamma} = \frac{q^5 - 1}{q^5},$$

$$\frac{\gamma_1}{\gamma^2} = \frac{1}{q^5}\cdot\frac{q^5 - 1}{q^5}\cdot\frac{t_1}{\pi \varepsilon},$$

$$\nu = 366{\cdot}26, \qquad \frac{\tau}{T} = 18{\cdot}6,$$

$$\nu' = 27{\cdot}32, \qquad \sigma = 70.$$

Also, taking the ellipticity which the earth would have had if it had been originally fluid and homogeneous, we have $\varepsilon = {\cdot}004$ nearly; which also gives

$$\frac{1}{\nu \varepsilon} = {\cdot}68.$$

Hence it appears that $\frac{n'}{\gamma}$ can never exceed $\frac{{\cdot}88}{18{\cdot}6}$ or ·047, a small quantity; but $\frac{n}{\gamma}$ may be greater than, equal to, or less than unity, according to the value of q, or the thickness of the earth's crust.

§. *Final Equations, giving the Numerical Values of x and y.*

We may now proceed to the substitution of the values found in the last section in the coefficients of equations E, F, G, and H (Art. 18.). I shall begin with equation E.

26. The coefficients of $\sin 2(nt + \lambda)$ is

$$\left\{-\frac{B}{2n} + \frac{\gamma_1}{\gamma^2}\frac{1}{1 - \left(\frac{n}{\gamma}\right)^2}\left(\frac{\gamma}{n}\frac{B}{D} - 1\right)\frac{D}{2}.\right.$$

Its value depends on that of $\frac{n}{\gamma}$, which may either be much less than unity, equal to unity, or greater than unity, according to the value of the ratio (q) which the outer bears to the inner radius of the shell (Art. 25.).

1. Let the shell be thin, or q nearly $= 1$; then will $\frac{n}{\gamma}$ be small (Art. 25.). Consequently the coefficient becomes

$$-\left(1-\frac{\gamma_1}{\gamma}\right)\frac{\text{B}}{2\,n}$$

$$= -\frac{\gamma_2}{\gamma}\cdot\frac{\text{B}}{2\,n}$$

$$= -\frac{q^5-1}{q^5}\cdot\frac{\text{B}}{2\,n}$$

$$= -\frac{3\,\varepsilon}{4\,\nu}\sin \text{I} \cos \text{I}.$$

2. Let the thickness of the shell be such that $\frac{n}{\gamma} = 1$ nearly. Then (Art. 25.)

$$2\,\frac{q^5-1}{q^5}\,\frac{1}{\nu\,\varepsilon} = 1 \text{ nearly},$$

$$q = \left(\frac{2}{2-\nu\,\varepsilon}\right)^{\frac{1}{5}} \text{ nearly},$$

and

$$= (3{\cdot}71)^{\frac{1}{5}} = 1{\cdot}3 \text{ nearly},$$

which determines the corresponding value of the ratio of the inner and outer radii of the shell. I shall reserve this case for a distinct consideration in the sequel.

3. Let $\frac{n}{\gamma}$ be greater than unity. If the shell be so thick that q becomes considerable, γ_1 will become small, and the coefficient will become

$$-\frac{\text{B}}{2\,n}$$

$$= -\frac{3\,\varepsilon}{4\,\nu}\sin \text{I} \cos \text{I}$$

(q being considerably greater than unity). This value is identical with that found in the former case.

The coefficient of $\sin 2\,(n'\,t + \lambda')$ $\left(\text{since } \frac{n'}{\gamma} \text{ is always small}\right)$ becomes

$$-\frac{\gamma_2}{\gamma}\,\frac{\text{B}'}{2\,n'} = -\frac{q^5-1}{q^5}\cdot\frac{\text{B}'}{2\,n'}$$

$$= -\frac{3}{4}\,\frac{\varepsilon}{\nu'(\sigma+1)}\cdot\frac{\tau}{\text{T}'}\cos 2\,\text{I} \sin 2\,i.$$

The coefficient of $\cos 2\,\gamma\,t$ consists of two parts, of which the latter $\left(\text{since } \frac{n'}{\gamma} \text{ is small}\right)$

reduces itself to

$$\frac{\gamma_1}{\gamma^2} \cdot \frac{D'}{2} \sin 2 \lambda' ;$$

and in like manner, when $\frac{n}{\gamma}$ is small, (or the thickness of the shell small) the first part becomes

$$\frac{\gamma_1}{\gamma^2} \cdot \frac{D}{2} \sin 2 \lambda,$$

and the whole coefficient becomes

$$= \frac{1}{q^5} \cdot \frac{3}{2} \left\{ \frac{\sin I}{\nu^2} \cdot \sin 2 \lambda - \frac{\sin I \sin 2 i}{2 \nu'^2 (\sigma + 1)} \sin 2 \lambda' \right\}.$$

The coefficient of $\sin 2 \gamma t$ becomes $\left(\text{if } \frac{n}{\gamma} \text{ be small}\right)$

$$- \left\{ \frac{\gamma_1}{\gamma^2} \cdot \frac{B}{2} \cos 2 \lambda + \frac{\gamma_1}{\gamma^2} \frac{B'}{2} \cos 2 \lambda' - \frac{\gamma_1}{\gamma^2} \frac{A}{2} \right\}$$

$$= - \frac{1}{q^5} \frac{3}{2} \cdot \left\{ \frac{\sin I \cos I}{\nu^2} \cdot \cos 2 \lambda + \frac{\cos 2 I \sin 2 i}{2 \nu'^2 (\sigma + 1)} \cos 2 \lambda' - \frac{\sin I \cos I}{\nu^2} - \frac{\sin I \cos I \cos^2 i}{\nu'^2 (\sigma + 1)} \right\}.$$

These two coefficients being affected with the factor $\frac{1}{q^5}$ are very much diminished when q becomes considerable.

The coefficient of $t = \frac{\gamma_2}{\gamma} A$, and becomes

$$\frac{q^5 - 1}{q^5} A$$

$$= \frac{3 \pi \varepsilon}{\nu} \sin I \cos I \left\{ \frac{1}{T} + \frac{1}{\sigma + 1} \cdot \frac{\nu}{\nu'} \cos^2 i \frac{1}{T'} \right\}.$$

The constant term becomes

$$\frac{3 \varepsilon}{4 \nu} \sin I \cos I \sin 2 \lambda + \frac{3 \varepsilon}{4 \nu' (\sigma + 1)} \cos 2 I \sin 2 i \frac{\tau}{T'} \sin 2 \lambda'.$$

Taking the expression for y, the coefficient of $\cos 2 (n t + \lambda)$ becomes, when $\frac{n}{\gamma}$ is small,

$$- \frac{\gamma_2}{\gamma} \frac{D}{2 n}$$

$$= - \frac{3 \varepsilon}{4 \nu} \sin I.$$

This is also true when q becomes considerable.

The coefficient of $\cos 2 (n' t + \lambda')$ becomes

$$- \frac{\gamma_2}{\gamma} \cdot \frac{D'}{2 n'}$$

$$= \frac{3 \varepsilon}{4 \nu' (\sigma + 1)} \cdot \cos I \sin 2 i \frac{\tau}{T'}.$$

The numerical values of the coefficients of $\sin 2\gamma t$ and $\cos 2\gamma t$ are respectively the same as those of $\cos 2\gamma t$ and $\sin 2\gamma t$ in the expression for x.

The constant term becomes

$$\frac{3\varepsilon}{4\nu}\sin I\cos 2\lambda - \frac{3\varepsilon}{4\nu'(\sigma+1)}\cos I\sin 2i\frac{\tau}{T'}\cos 2\lambda'$$
$$+\frac{1}{q^5}\left\{\frac{3}{2\nu^2}\sin I\cos I + \frac{3}{2\nu'^2(\sigma+1)}\cos^2 i\right\}.$$

27. Hence we obtain the following expressions for x and y, for any thickness of the shell for which $\frac{n}{\gamma}$ is small.

$$x = -\frac{3\varepsilon}{4\nu}\sin I\cos I\sin 2(nt+\lambda) - \frac{3}{4}\frac{\varepsilon}{\nu'(\sigma+1)}\cdot\frac{\tau}{T'}\cos 2I\sin 2i\sin 2(n't+\lambda')$$
$$+\frac{1}{q^5}\cdot\frac{3}{2}\left\{\frac{\sin I}{\nu^2}\sin 2\lambda - \frac{\sin I\sin 2i}{2\nu'^2(\sigma+1)}\sin 2\lambda'\right\}\cos 2\frac{q^5}{q^5-1}\pi\varepsilon\frac{t}{t_1} \qquad \text{(I.)}$$
$$-\frac{1}{q^5}\cdot\frac{3}{2}\left[\begin{matrix}\frac{\sin I\cos I}{\nu^2}\cos 2\lambda + \frac{\cos 2I\sin 2i}{2\nu'^2(\sigma+1)}\cos 2\lambda' \\ -\frac{\sin I\cos I}{\nu^2} - \frac{\sin I\cos I\cos^2 i}{\nu'^2(\sigma+1)}\end{matrix}\right]\sin 2\frac{q^5}{q^5-1}\pi\varepsilon\frac{t}{t_1}$$
$$+\frac{3\pi\varepsilon}{\nu}\sin I\cos I\left\{\frac{t}{T} + \frac{1}{\sigma+1}\cdot\frac{\nu}{\nu'}\cos^2 i\frac{t}{T'}\right\} + C;$$

$$y = -\frac{3\varepsilon}{4\nu}\sin I\cos 2(nt+\lambda) + \frac{3\varepsilon}{4\nu'(\sigma+1)}\cos I\sin 2i\frac{\tau}{T'}\cos 2(n't+\lambda')$$
$$+\frac{1}{q^5}\cdot\frac{3}{2}\left\{\frac{\sin I}{\nu^2}\sin 2\lambda - \frac{\sin I\sin 2i}{2\nu'^2(\sigma+1)}\sin 2\lambda'\right\}\sin 2\frac{q^5}{q^5-1}\pi\varepsilon\frac{t}{t_1} \qquad \text{(K.)}$$
$$+\frac{1}{q^5}\cdot\frac{3}{2}\left[\begin{matrix}\frac{\sin I\cos I}{\nu^2}\cos 2\lambda + \frac{\cos 2I\sin 2i}{2\nu'^2(\sigma+1)}\cos 2\lambda' \\ -\frac{\sin I\cos I}{\nu^2} - \frac{\sin I\cos I\cos^2 i}{\nu'^2(\sigma+1)}\end{matrix}\right]\cos .\, 2\frac{q^5}{q^5-1}\pi\varepsilon\frac{t}{t_1} + C';$$

when C and C′ are small constant terms whose values are given above.

These are the expressions for x and y when $\frac{n}{\gamma}$ is small, or the thickness of the shell comparatively small. When the thickness is such that q becomes considerably greater than unity, the terms involving sine and cosine of $2\frac{q^5}{q^5-1}\pi\varepsilon\frac{t}{t_1}$ may be entirely omitted, and the expressions will then be true in this case.

28. Since the motion of the interior fluid cannot be subjected to observation, it would be useless to make the substitution of numerical values in equations (G.) and (H.) (Art. 18.). We may remark, however, that the motion of the axis of instantaneous rotation of the fluid will be exactly similar to that of the axis of the shell, and of the same order, as is easily seen by comparing the two equations just mentioned with the equations (E) and (F) of the same article.

§. *Interpretation of the Final Expressions for x and y* (Art. 27).

29. The terms in x and y which have $2\,(n\,t + \lambda)$ for their arguments are the two parts of solar nutation. They are identical with the expressions for solar nutation deduced on the hypothesis of the earth's being a homogeneous solid spheroid. It will be recollected that this excludes the particular case in which the outer and inner radii are in a certain ratio to each other (Art. 26. 2.).

The terms of which the argument is $2\,(n'\,t + \lambda')$ are the two parts of lunar nutation, which are, for any thickness of the shell, identical with the expressions deduced on the hypothesis of the earth's being entirely solid and homogeneous.

The term in x which constantly increases with t is the luni-solar precession. It is again identical with that found on the hypothesis just mentioned. We may also remark, that this agreement is independent of any approximation depending on the smallness of such quantities as $\frac{n}{\gamma}$ or $\frac{n'}{\gamma}$, and is consequently more accurately true than in the expressions for nutation.

30. The terms of which the common argument is $2\,\frac{q^5}{q^5 - 1}\,\pi\,\varepsilon\,\frac{t}{t_1}$ or $2\,\gamma\,t$ indicate an inequality depending entirely on the fluidity of the interior mass. If we denote the coefficients in these terms by G and H, and neglect the other terms, we shall have

$$x = \mathrm{G}\cos 2\,\gamma\,t - \mathrm{H}\sin 2\,\gamma\,t,$$
$$y = \mathrm{G}\sin 2\,\gamma\,t + \mathrm{H}\cos 2\,\gamma\,t;$$

or

$$x = \sqrt{\mathrm{G}^2 + \mathrm{H}^2}\cos 2\,(\gamma\,t + \mathrm{K}),$$
$$y = \sqrt{\mathrm{G}^2 + \mathrm{H}^2}\sin 2\,(\gamma\,t + \mathrm{K});$$

$\left(\text{where } \tan\ 2\mathrm{K} = \frac{\mathrm{H}}{\mathrm{G}}\right)$ which show that x and y would thus be the coordinates of a point moving uniformly in a circle; and if R be its radius

$$\mathrm{R} = \sqrt{\mathrm{G}^2 + \mathrm{H}^2};$$

and the period of revolution would $= \frac{\pi}{\gamma}$

$$= \frac{q^5 - 1}{q^5}\,\frac{t_1}{\epsilon}.$$

It appears by the expressions for G and H, that these quantities will be the greatest when q^5 is least, i. e. when the shell is very thin; but even in that case they will not rise to magnitudes greater than those of the order of the solar nutation; and when the thickness of the shell becomes considerable, and q differs considerably from unity, the inequality will become quite insensible.

There is a corresponding inequality in the motion of the axis of instantaneous rotation of the fluid, indicated by corresponding terms in x' and y'. Comparing them with the terms in x and y, we find (omitting the other terms)

$$x' = -\frac{\gamma_2}{\gamma_1} \text{G} \cos 2\gamma t + \frac{\gamma_2}{\gamma_1} \text{H} . \sin 2\gamma t,$$

$$y' = -\frac{\gamma_2}{\gamma_1} \text{G} \sin 2\gamma t - \frac{\gamma_2}{\gamma_1} \text{H} \cos 2\gamma t;$$

or

$$x' = -\frac{\gamma_2}{\gamma_1} \sqrt{\text{G}^2 + \text{H}^2} \cos 2(\gamma t + \text{K}),$$

$$y' = -\frac{\gamma_2}{\gamma_1} \sqrt{\text{G}^2 + \text{H}^2} \sin 2(\gamma t + \text{K}).$$

Consequently the locus of $x'\,y'$ would also be a circle described about the common origin of $x\,y$, x' and y', and having a radius $= \frac{\gamma_2}{\gamma_1}$ R. By this inequality, therefore, alone the points P and P′ would describe circles about the same centre in the same periodic time, with radii in the ratio of $\gamma_1 : \gamma_2$, and differing in angular position by 180°.

The motion now described is that which would obtain if no extraneous disturbing forces acted on the spheroid, and the axes of instantaneous rotation of the shell and fluid should be separated by a small angle. It is a case of rotatory motion which has not before been investigated.

31. The case which remains for our consideration is that in which $\gamma = n$ nearly (Art. 26.).

In our previous investigations we have supposed the spheroidal shell to be of a definite constant thickness, and not to increase with the time. In the case of the earth, however, in which the solidity of the shell is conceived to be due to the external refrigeration of the mass, this thickness must be constantly increasing, though the rate of increase must be excessively slow; and our results, as expressed in equations (E), (F), (G), and (H) (Art. 18.), will be true for any instantaneous value of q, or of the thickness of the shell. So far, however, as the inequalities are of appreciable magnitude, we have seen that they are independent of particular values of q, except in the case which we have now to consider.

Referring to equation (E) (Art. 18.) we find that when $\gamma - n$ is very small, we have (taking what then become the most important periodical terms)

$$x = \frac{\gamma_1}{\gamma^2 - n^2}\left(\frac{\gamma}{n}\frac{\text{B}}{\text{D}} - 1\right)\frac{\text{D}}{2}\sin 2(n t + \lambda)$$

$$- \frac{\gamma_1}{\gamma^2 - n^2}\left(\frac{\gamma}{n}\frac{\text{B}}{\text{D}} - 1\right)\frac{\text{D}}{2}\sin 2\lambda \cos 2\gamma t$$

$$+ \frac{\gamma_1}{\gamma^2 - n^2}\left(\frac{\gamma}{n} - \frac{\text{B}}{\text{D}}\right)\frac{\text{D}}{2}\cos 2\lambda \sin 2\gamma t.$$

Now $\frac{\gamma_1}{\gamma^2 - n^2} = \frac{\gamma_1}{\gamma + n} \cdot \frac{1}{\gamma - n} = \frac{\gamma_1}{2n} \cdot \frac{1}{\gamma - n}$ (since $\gamma = n$ nearly);

also

$$\frac{\text{B}}{\text{D}} = \cos \text{I}.$$

Hence putting $\frac{\gamma_1}{2n}(1 - \cos I)\frac{D}{2} = h$, we have

$$x = -\frac{h}{\gamma - n}\sin 2(n t + \lambda)$$

$$+\frac{h}{\gamma - n}\left\{\sin 2\lambda \cos 2\gamma t + \cos 2\lambda \sin 2\gamma t\right\}$$

$$= \frac{h}{\gamma - n}\left\{\sin 2(\gamma t + \lambda) - \sin 2(n t + \lambda)\right\}$$

$$= 2h . \frac{\sin 2\overline{(\gamma - n)t}}{\gamma - n}\cos 2(n t + \lambda) \text{ very nearly,}$$

an expression which assumes the form $\frac{0}{0}$ when $\gamma = n$ accurately. To put it under a more convenient form, assume t' a particular value of t, such that

$$2(\gamma - n)t' = \text{some multiple of } \pi$$
$$= 2m\pi;$$

and let $t = t' + t''$,

$$\gamma - n = s,$$

and when $t = t'$,

let $\gamma - n = s_1$.

For the clearer interpretation of this term, let us first suppose the thickness of the shell, and therefore γ and s, to remain constant. We have

$$x = 2h\frac{\sin 2 s_1 (t' + t'')}{s_1}\cos 2(n\overline{t' + t''} + \lambda)$$

$$= \frac{2h}{s_1}\sin 2 s_1 t'' \cos 2(n t'' + L).$$

In a similar manner we obtain

$$y = \frac{2h}{s_1}\sin 2 s_1 t'' . \sin 2(n t'' + L).$$

Since s_1 is supposed very small compared with n and the product $s_1 t''$ may be considered as nearly constant for any one year, in which time $\sin 2(n t'' + L)$ will pass through its period, and the solar nutation for that year will depend on

$$\frac{2h}{s_1}\sin 2 s_1 t''.$$

Consequently from the time when $t = t'$ or $t'' = 0$, this nutation will increase every year till $2 s t'' = \frac{\pi}{2}$, after which it will again decrease. We should thus have a *secular inequality* in the solar nutation, of which the whole period would be $\frac{\pi}{s_1}$, and of which the greatest value, with reference either to x or y, would be $\frac{2h}{s_1}$.

In the actual case in which γ constantly decreases, suppose that at the time t'' from the time t',

$$s = \gamma - n = s_1 - r\,t'',$$

r denoting the rate of decrease of γ, and being taken constant during a small augmentation in the thickness of the shell. Then shall we have

$$x = \frac{2h}{s_1 - r\,t''} \sin \overline{2\,(s_1 - r\,t'')\,t''} \,.\cos 2\,(n\,t + \mathrm{L}),$$

with a similar expression for y.

This expression indicates a *secular variation* in the *secular inequality* just noticed, which increases with the diminution of $s_1 - r\,t''$, or the increase of t'', till $r\,t''$ becomes greater than s_1, after which the inequality will constantly decrease again.

The determination of r would require that of the rate at which the thickness of shell may increase. We have

$$\gamma - n = s = s_1 - \frac{d\,s}{d\,t}\,t''$$

$$= s_1 - \frac{d\,\gamma}{d\,t}\,t'',$$

$$\therefore r = \frac{d\,\gamma}{d\,t}.$$

But

$$\gamma = \frac{q^5}{q^5 - 1}\,\pi\,\varepsilon\,\frac{1}{t_1} \qquad \text{(Art. 25.)}$$

$$= \frac{a_1^{\,5}}{a_1^{\,5} - a^5}\,\frac{\pi\,\varepsilon}{t_1};$$

$$\frac{d\,\gamma}{d\,t} = 5\,\frac{a_1^{\,5}\,a^4}{(a_1^{\,5} - a^5)^2}\,\frac{\pi\,\varepsilon}{t_1}\cdot\frac{d\,a}{a\,t}$$

$$= 5\,\frac{q^5}{(q^5 - 1)^2}\,\frac{\pi\,\varepsilon}{t_1}\cdot\frac{1}{a}\cdot\frac{d\,a}{a\,t}.$$

Let $\delta\,a$ = increase of thickness in time T (one year); then

$$\delta\,a = \frac{d\,a}{d\,t}\,\mathrm{T},$$

and

$$r = 5\,\frac{q^5}{(q^5 - 1)^2}\,\frac{\pi\,\varepsilon}{\mathrm{T}\,t_1}\cdot\frac{\delta\,a}{a}$$

$$= \cdot 35\,\frac{\delta\,a}{a}\cdot\frac{1}{\mathrm{T}\,t_1}$$

by the substitution of numerical values (Art. 26. 2.). Hence r may be known when $\frac{\delta\,a}{a}$ is determined.

Substituting for γ_1 its value $\frac{\pi\,\varepsilon}{q^5 - 1}\,\frac{1}{t_1}$, and for n its value $\frac{2\,\pi}{\mathrm{T}}$, we have

$$h = \frac{\nu\,\varepsilon}{8}\cdot\frac{1}{q^5 - 1}\cdot(1 - \cos \mathrm{I})\,\mathrm{D}.$$

We may here, without sensible error, put for q its particular value $\left\{\left(\frac{2}{2 - \nu \varepsilon}\right)^{\frac{1}{5}} \text{(Art. 26. 2.)}\right\}$ when $\gamma = n$. We thus obtain

$$h = \cdot 02 \frac{\pi}{\mathrm{T}} \text{ nearly, } 1'' \text{ being the angular unit.}$$

Also (Art. 25.)

$$\gamma - n = \frac{q^5}{q^5 - 1} \pi \varepsilon \frac{1}{t_1} - \frac{2\pi}{\mathrm{T}},$$

$$= \left(\frac{q^5}{q^5 - 1} \nu \varepsilon - 2\right) \frac{\pi}{\mathrm{T}},$$

$$= \left(\frac{q^5}{q^5 - 1} 1 \cdot 464 - 2\right) \frac{\pi}{\mathrm{T}}.$$

Hence the greatest value of the secular inequality $= \frac{2h}{\gamma - n}$,

$$= \left(\frac{\cdot 04}{\frac{q^5}{q^5 - 1} 1 \cdot 464 - 2}\right)'';$$

and the whole period of the inequality $= \frac{\pi}{\gamma - n}$,

$$= \frac{\mathrm{T}}{\frac{q^5}{q^5 - 1} 1 \cdot 464 - 2}.$$

If we assign any particular value to the above expression for $\frac{2h}{\gamma - n}$, we may easily determine the corresponding value of q, and of the thickness of the solid shell, and also the period of the inequality, supposing the thickness of the shell to remain very nearly the same during such period. Thus, suppose the greatest value of the inequality to be $5''$, we shall have

$$\frac{\cdot 04}{\frac{q^5}{q^5 - 1} 1 \cdot 464 - 2} = 5.$$

This gives

$$\frac{1}{q} = \cdot 77 \text{ nearly,}$$

or

$$a = \cdot 77\, a_1.$$

Also when $\gamma = n$ accurately, we have (if q' be the corresponding value of q)

$$\frac{q'^5}{q'^5 - 1} 1 \cdot 464 - 2 = 0,$$

which gives

$$\frac{1}{q'} = \cdot 768472,$$

or, if a' be the corresponding value of a,

$$a' = \cdot 768472 \,.\, a_1.$$

Therefore

$$a - a' = 6 \text{ miles approximately.}$$

The period would be about 125 years.

In order that these numerical results may be approximately true, the variation of $a - a'$ during the period of 125 years must be small compared with $a - a'$. If we suppose the thickness of our shell to increase at the same rate as that of the earth's crust in the process of its solidification, this will probably be true.

Again, if we suppose the inequality to amount to about 1000″, we obtain

$$a - a' = 130 \text{ feet nearly};$$

and the period, supposing a constant, would be about 25,000 years, i. e. in one fourth of that period the part of solar nutation dependent on the term we are discussing would pass from zero to about 1000 seconds. If, however, we suppose the solid shell of our spheroid to increase in thickness at the same rate as the crust of the earth, the difference between a and a' would possibly not remain within the value just mentioned for nearly so long as 6000 years; in which case, supposing the inequality to be zero when $a - a'$ should equal about 130 feet, it could never afterwards amount to nearly 1000 seconds; nor could it have been previously so great, because its previous values must have corresponded to values of $a - a'$ less than the above value. Our investigation, however, does not tell us whether 120 or 130 feet would be near the value of $a - a'$ the last time the secular inequality should vanish before a became $= a'$, and consequently we cannot say with certainty that 1000 seconds would be the extreme limit to which the inequality would attain. In fact, the exact determination of this limit would require the very accurate determination of a as a function of t, which cannot be known in the case of the earth's crust without an accurate knowledge of the conductive power of the matter which constitutes it. From the small value, however, of $a - a'$ and great length of the period corresponding to the maximum of 1000″ for the inequality we have been considering, it may perhaps be deemed extremely improbable that it should ever exceed that value in the case of the earth. The duration of time for which the effect of the cause we are discussing on solar nutation would be sensible to observation would be, that necessary for the thickening of the earth's crust so to increase that $a - a'$ should pass from $+$ (6 or 8 miles) to $-$ (6 or 8 miles), and might therefore be approximately determined if the quantity denoted by r in this article were known.

§. *Degree of Approximation in the preceding Results.*

The results at which we have arrived above rest on the hypothesis of the instantaneous planes of rotatory motion being parallel to the tangent plane to the interior surface of the shell at B′ (Art. 8. III.); and it remains for us to consider the degree of approximation to the actual motion which has been thus obtained. It will be recollected that, on this hypothesis, the centrifugal force produces a force $Z = 2\,\omega^2\,\varepsilon\,\beta\,.\,x$ (Art. 10.), which alone is effective in producing motion in the fluid, this motion being about the axis of y, and that which, combined with the angular motion about A B′, causes the angular motion in space of this latter axis. The value of **Z** has been found on the hypothesis of ω being constant, or of the rotatory motion about A B′ being the same as if the sections of the inner surface of the shell were circles instead of being ellipses of small eccentricity (Art. 9.); and the pressure on the inner surface of the shell depending on the centrifugal force has been calculated on the same hypothesis. It will be necessary therefore to examine the errors thus committed.

33. Let us conceive a closed cylinder entirely filled with fluid, which revolves uniformly about the axis of the cylinder with a velocity ω, and is not acted on by any external force. If the form of the cylinder be then changed without changing its volume, so that each section perpendicular to its axis shall become an ellipse of small eccentricity instead of being circular, it is manifest from the conditions of symmetry, that the angular motion of the fluid, though no longer uniform, will still be *steady* about the same axis, as in the circular cylinder. Consequently if p' be the pressure at any point on the surface of the elliptical cylinder, v' the velocity of the fluid at that point, we shall have

$$p' = \mathrm{C} - \frac{1}{2} v'^2;$$

and if p and v be any corresponding values of p' and v' (which may be taken for their mean values)

$$p = \mathrm{C} - \frac{1}{2} v^2,$$

and

$$p' = p - \frac{1}{2} (v'^2 - v^2).$$

Now the quantity of fluid which passes through any section of the elliptic cylinder made by a plane through its axis, must be constant, and therefore the velocity v' must vary inversely as r'*, the radius vector of the elliptic section from its centre. Therefore, if r be the value of r' when v is that of v' (i. e. the mean value of r')

$$v'^2 = v^2 \cdot \frac{r^2}{r'^2},$$

$$= v^2 \cdot \frac{r^2}{b'^2} (1 - 2\,\varepsilon' \cos \theta'),$$

where b' is the axis minor of the elliptic section. Also

$$r = \frac{a' + b'}{2},$$

$$= b'\left(1 + \frac{\varepsilon'}{2}\right);$$

and

$$\frac{r^2}{b'^2} = 1 + \varepsilon':$$

* There can be no doubt of this hypothesis being true, at least to a sufficient degree of approximation for our immediate purpose.

whence

$$v'^2 - v^2 . = v^2 \varepsilon' (1 - 2 \cos^2 \theta') ;$$

and

$$p' = p + \frac{1}{2} v^2 \varepsilon' \cos 2 \theta',$$

or, putting in the small term ωr for v,

$$p' = p + \frac{\omega^2}{2} r^2 \varepsilon' \cos 2 \theta'.$$

We have here taken r' and θ' as the polar coordinates of the elliptical section of the surface, but it is evident that this expression for the fluid pressure will be equally true for any point of the fluid of which r' and θ' are the co-ordinates, p and r being taken with reference to an ellipse passing through the point $(r' \theta')$ similar to the elliptic section of the cylinder, and similarly situated.

The case which presents itself in our actual problem is analogous to the one just considered, so far as regards the elliptical form of the sections made by the planes of rotatory motion, these planes being parallel to the tangent plane at B′ (fig. 2.). The common ellipticity of these sections is $\varepsilon \beta$ (Art. 9.), and, therefore, in finding the effect on the shell, of the pressure arising from the centrifugal force on the fluid in article 23., we ought to have used for p the pressure as found in the last article, p', or

$$p + \frac{\omega^2}{2} r^2 \varepsilon \beta \cos 2 \theta'.$$

This, however, would only introduce into p a term of the order $\varepsilon \beta$, and which, therefore, may be omitted, as shown in the investigation just referred to (Art. 23.). Also, taking this expression for p' as applicable to any point of the fluid (as explained in the preceding paragraph), it is easily seen that the force Z $(= \omega^2 \varepsilon \beta . x)$ will only be altered, in consequence of the ellipticity of the sections of the shell, by a quantity small compared with itself, and which may, therefore, be neglected. Our results, then, will be quite accurate to the degree of approximation to which we have proceeded, assuming the parallelism of the planes of rotatory motion to the tangent plane at B′. I shall now proceed to this point.

34. It has been shown (Art. 15.) that the angular velocity of A B′ in space $= \omega . \varepsilon \beta$, and also (Art. 28.) that the angular velocity of A B is of the same order as that A B′ i. e. of the order $\varepsilon \beta$. The angular motion of A B′ (Arts. 12 ..... 15.) is due entirely to the *obliquity* of the planes of rotatory motion of the fluid particles, and the above value of it is calculated on the hypothesis of these planes of rotation being parallel to the tangent plane at B′. It is easy to see, however, that if this hypothesis be not accurate, the value of γ_2 (Art. 16.) and therefore of the angular velocity of A B′ (Art. 15.) will still be quantities of the same order respectively as the calculated values which have been given, so long as the planes of rotation shall make angles with planes perpendicular to A B′ which, instead of being = to ι (Art. 8.), shall be merely of the same order of magnitude. Without assuming, therefore, the accuracy of the above

hypothesis, we may still assert that the angular velocities of A B and A B′ will necessarily be of the order $\varepsilon\ \beta$.

The positions of the planes of rotatory motion will be affected by the change of position of the shell, or of A B, and also by that of A B′. It will be convenient to consider these cases separately, first, supposing A B′ fixed while A B moves, and then taking A B fixed and A B′ in motion.

In the assumption, that A B′ shall be at rest, it is meant that it shall here be considered as unaffected by the angular motion which has been investigated, and which is due to the obliquity of the planes of rotatory motion. Our first object will be to examine whether any motion will be communicated to A B′ as the direct and immediate consequence of the motion of the shell, and independently of centrifugal force in the fluid, to which the previously calculated velocity of A B′ is entirely due; also A B′ ought strictly to be considered as *the line of quiescent fluid particles*, in which sense it will not necessarily be a straight line, as we have hitherto considered it in calculating the effects of centrifugal force on the fluid. It will, therefore, be necessary to examine the degree of its deviation from rectilinearity.

Suppose the shell to be at first in the position represented by the dotted line (fig. 2.) and then to be brought into that represented by the continuous line, A B, coinciding at first with A B′. Then while A B moves through the angle B′ A B (β), the normal motion (N N″) at any point (N) cannot exceed a quantity of the order $\varepsilon\,\beta$, as is easily shown*. Also it is evident that (considering only the velocity due to the displacement of the shell) the ratio

$$\frac{\text{vel. of fluid particle at N}}{\text{vel. of point B of the shell}}$$

must be a quantity of the same order as $\frac{\text{N N}''}{\text{B}'\text{ B}}$, i. e. of the order ε; and it is easily seen that for any particle in the interior of the fluid the motion cannot exceed a quantity of that order. Also the conditions of symmetry will evidently require that the particle at A should remain at rest.

If the spheroidal axis, instead of moving from A B′ to A B, move from A B to A B, the same conclusion respecting the ratio of the velocity of any fluid particle to the velocity of B will still be true, as is easily seen.

Let v_1 be the velocity of B, v that of a particle Q, from the cause we are considering, the distance of Q from A being r, and its distance from the axis A B′ $= \varrho$. Since v will be of the order $\varepsilon\ v_1$ let $v = k\ \varepsilon\ \frac{r}{a}\ v_1$, where k is a numerical quantity, the value of which may depend on the position of the particle. Also the velocity of Q from the motion of rotation round A B′ $= \omega\,\varrho$. Consequently, if Q be so situated that these

* N ν must manifestly vanish with ϵ as well as with β, and the expression for it must, therefore, involve some power of ϵ as a factor.

velocities are impressed upon it along the same line and in opposite directions, the whole velocity of Q will

$$= k \varepsilon \frac{r}{a} v_1 - \omega \varrho,$$

and if this $= 0$, Q will be a point in the line of quiescent particles. This gives us

$$\frac{\varrho}{r} = \frac{k \varepsilon}{\omega} \cdot \frac{v_1}{a} :$$

$\frac{v_1}{a}$ is the angular velocity in space of A B, and is, therefore, of the order $\varepsilon \beta$, and $= k' \varepsilon \beta$ (suppose). Consequently, the angular deviation of Q from A B' $\left(\text{which} = \frac{\varrho}{r}\right)$

$$= \frac{k k'}{\omega} \cdot \varepsilon^2 \beta,$$

a quantity extremely small compared with β. A B' may, therefore, be considered as a straight line, to the required degree of approximation. Also the angular velocity of any point in A B' due to the cause here considered, is of the order $\varepsilon^2 \beta$; it may, therefore, be neglected in comparison with the angular velocity ($\omega \varepsilon \beta$) of A B' previously determined (Art. 15.).

35. We may now proceed to consider the positions of the planes of rotation of different particles of the fluid when B does not coincide with B'. It has been shown (Art. 8. III.) that the instantaneous positions of these planes must approximate more or less to parallelism with the tangent plane at B'. This approximation, however, may be different for different fluid particles, in which case it will manifestly be most accurate for particles nearest to B' and b', and less so for those nearer the plane of the equator. In considering, therefore, the degree of approximation it may be convenient to refer to a *mean plane* of rotation, or an imaginary plane whose inclination is the mean of the inclinations of all the planes of rotation of different particles.

As B moves about B' the tangent plane at B' will move from one position to another, revolving about its ultimate intersection with the consecutive position, as an axis of instantaneous rotation, with a certain angular velocity. If B moved uniformly round B' (as would be the case if the motion were due entirely to the centrifugal force on the fluid (Art. 30.)), the angular velocity of the tangent plane would be uniform; and since the motion of the fluid would then be steady, the angular motion of the planes of rotation would also be uniform, and the angle thus described in a unit of time by the mean plane of rotation might be taken as a measure of the whole *constraining force* (arising from the reaction of the solid shell on the fluid) which produces this particular motion. In the actual case the motion of B will not be uniform; but since the *variation* of its motion will be extremely slow, the propositions just enunciated will still be approximately true for any comparatively limited period, and we may still take as a measure of the instantaneous *constraining force*, the angle actually described by the mean plane of rotation, in the manner above explained, in a unit of time.

Now if A B should move from A B′ through an angle β, it is easy to show that the tangent plane at B′ must move through an angle of the order $\varepsilon\,\beta$; and it is easily seen likewise that if A B move in any other direction, as from A B to A 𝔅 through an angle β', the angle moved through by the tangent plane at B′ will be of the order $\varepsilon\,\beta'$. In every case, therefore,

$$\begin{aligned}\text{ang. vel. of the tang. plane} &= k'\,\varepsilon\,.\,\text{ang. vel. of A B}\\ &= k\,\varepsilon^2\,\beta\end{aligned}$$

where k is some finite numerical quantity. Consequently, since the angle described by the mean plane of rotation in a unit of time cannot be greater than this, that angle, and therefore the instantaneous *constraining force*, must be of the order $\varepsilon^2\,\beta$.

36. Let us now consider the relation between the *constraining force* and the angle which the instantaneous mean plane of rotation makes with the instantaneous tangent plane at B′. Let ι denote, as heretofore, the angle between the tangent plane and a plane perpendicular to A B′, ι' that between the tangent plane and the mean plane. Now instead of the shell moving on continuously, let us conceive its motion to cease at any instant, and consider its action, when thus at rest, on the fluid mass. If we take a fluid particle near to B′ and in contact with the surface, B′ may be considered as the centre of its rotatory motion, provided its distance from that point be not less than a quantity of the order $\varepsilon\,\beta$ (since the angular displacement of B′ cannot exceed a quantity of the order $\varepsilon^2\,\beta$). Consequently, if the motion of the plane should cease at any instant, as above supposed, it is manifest that the plane of motion of this particle must be immediately constrained to coincide with the tangent plane at B′, i. e. the constraining force upon it must have been such as to change its plane of motion through an angle of the order ι' in a very short space of time. If we take a particle in contact with the surface, rather more remote from B′, the same conclusion must be approximately true, though a somewhat greater time may be necessary to produce an equal change in the position of its plane of rotation. And similarly if we take a particle in contact with the shell at any point, for instance, between B′ and N′, the reaction of the surface must produce a similar effect on its plane of rotation; and moreover, it is easily seen that if the shell be supposed to remain thus at rest for a whole revolution, for example, the effect produced in that time must be of the same order of magnitude as that for particles near to B′. Precisely the same effects must take place about b' and between b' and L′, whence it will necessarily follow that similar effects and of the same order of magnitude must be produced on the planes of motion of the particles constituting the interior part of the fluid intermediate to the portions N′ n' and L′ l' of the surface. Similar effects must also be produced on the portion of the fluid exterior to that just specified, though these effects may decrease in magnitude as the particles are situated nearer to C and c.

Hence then it follows that (taking, for the greater distinctness, one day for the unit of time) if B, and therefore the tangent plane at B′, were to remain at rest for a unit of time, the *constraining force*, estimated as above described, arising from the reaction of the shell on the fluid, would be such as to cause the mean plane of rotation to move through an angle of the same order of magnitude as the instantaneous angle between that plane and the tangent plane at B′, i. e. of the order ι'. But it is evident that if the tangent plane at B′, instead of remaining at rest, as we have here conceived it to do, have its actual motion *from* the instantaneous mean plane, the whole effect in one day on the plane of rotation of a particle near to B′ or b' must be greater than if the surface had remained at rest; and the same conclusion must also be true for particles more remote from B′ or b'. Consequently the angle through which the mean plane of rotation moves in one unit of time, must, *à fortiori*, in the actual motion be of the same order as ι', i. e. the *constraining force*, estimated by this angle, must be of the order ι'. But it has been already shown (Art. 35.) that this force must be of the order $\varepsilon^2 \beta$. Consequently ι' must be of the order $\varepsilon^2 \beta$; or, since $\iota = 2 \varepsilon \beta$, the angle between the mean plane of rotation and the tangent plane at B′ is a small quantity of a higher order than ι, which proves the truth of our assumption, in the previous investigations, of the coincidence of these planes to the required degree of approximation.

37. We have hitherto considered B to move while B′ remains at rest; let us now consider B′ to move while B remains at rest. Suppose A B′ to move through an angle β' in its motion in space which has been previously investigated, B′ then coming to 𝔅′. If the shell were spherical, the angle between the tangent planes at B′ and 𝔅′ respectively would $= \beta'$, and in the spheroid the angle between these planes can differ from β' only by a quantity of the order $\varepsilon \beta'$. Consequently, in order that the mean plane of rotation should be always parallel to the tangent plane at the extremity of the axis of rotation of the fluid, it must move through an angle of the same order as that (β') described by that axis; whereas when A B moves through an angle β', the corresponding angular motion of the mean plane of rotation is (as we have shown) only of the order $\varepsilon \beta'$. We must examine how this angular motion of the mean plane is produced when A B′ is in motion.

While the axis of instantaneous rotation in a rigid body changes its position in the body, the instantaneous planes of rotatory motion necessarily retain their perpendicularity to it, and therefore the angular motion of those planes is equal to that of the axis. Now we have shown (Art. 15.) that the change in the position of A B′ is produced in a manner exactly similar to that in a solid body, so that the same cause produces simultaneously the angular motion of A B′, and an equal angular motion of the planes of rotation; whence it is easily seen that the mean plane of rotatory motion when the axis has moved to 𝔅′, cannot, on this account alone, deviate from parallelism with the tangent plane at 𝔅′ by a quantity greater than of the order $\varepsilon \beta'$. Consequently the additional angular velocity of the mean plane of motion necessary to preserve it in parallelism with the tangent plane cannot exceed a quantity of the order

$$\varepsilon \,.\, \text{ang. vel. of A B}',$$

$$= \text{K}\, \varepsilon^2 \beta.$$

This additional angular velocity must be produced by the *constraining force* as previously described. The force, therefore, in this case, as well as in the one previously considered, must be of the order $\varepsilon^2 \beta$; whence it also follows, as before, that the angle between the instantaneous mean plane and the tangent plane at the extremity of the axis of rotatory motion of the fluid must be of the order $\varepsilon^2 \beta$, a quantity to be neglected in comparison with ι.

Also, since ι' is small compared with ι in each of the above cases considered independently, it will be likewise true when the two causes act simultaneously, i. e. in the actual motion of B and B′ about each other. Hence all our previous results will be true to the required degree of approximation.

The following then are the results at which we have arrived, supposing the earth to consist of a homogeneous spheroidal shell (the ellipticities of the outer and inner surfaces being the same) filled with a fluid mass of the same uniform density as the shell.

I. The precession will be the same, whatever be the thickness of the shell, as if the whole earth were homogeneous and solid.

II. The lunar nutation will be the same as for the homogeneous spheroid to such a degree of approximation that the difference is inappreciable to observation.

III. The solar nutation will be sensibly the same as for the homogeneous spheroid, unless the thickness of the shell be very nearly of a certain value, something less than one-fourth of the earth's radius, in which case this nutation might become much greater than for the solid spheroid.

IV. In addition to the above motions of precession and nutation, the pole of the earth would have a small circular motion, depending entirely on the internal fluidity. The radius of the circle thus described would be greatest when the thickness of the shell should be least; but the inequality thus produced would not for the smallest thickness of the shell exceed a quantity of the same order as the solar nutation; and for any but the most inconsiderable thickness of the shell would be entirely inappreciable to observation.

In my next communication I propose to consider the case in which both the solid shell and the inclosed fluid mass are of variable density.

W. Hopkins.

Cambridge,
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